Derivative Securities: The Atomic Structure*

Jerome Kreuser**

Lester Seigel***

Abstract

In this paper, derivative security building blocks of constant risk (omega) are identified which satisfy the Black and Scholes equation and which, in aggregate, can be combined to form a wide class of derivative structures; once in place the elemental options, which we term "atoms", need only modest rebalancing. A user-friendly technology based on Legendre techniques is introduced to ascertain the appropriate combination of derivative atoms needed to assemble more complex risk profiles. Extension of this solution technique to a multifactor world is outlined, with Robert Merton's valuation of options incorporating stochastic interest rates used as an illustration.

1. *Introduction: The Engineering of Things*

Most of us view engineering as the building of things from smaller things. Furthermore, once the smaller things are in place, they are relatively invariant – there is no dynamic adjustment of the parts to maintain the whole. Nature, the master engineer, also appears to be quite satisfied with this scheme; there apparently is no need to continuously change the proportion of hydrogen and oxygen atoms in maintaining a water molecule. Yet, when we turn to "Financial Engineering", the state of play changes dramatically. The building blocks are often combinations of small things and bigger, complex things and the proportion of the components varies substantially over the lifetime of the structure synthesized. For example, the hedging of stock options involves the dynamic rebalancing of cash, stock, and often more complex instruments with appropriate nonlinear behavior. In brief, financial engineering conveys only a limited sense of putting together elemental stable parts into building a coherent whole.

This also has implications for the teaching of Derivative Securities at the graduate level. In particular, it is common to carefully develop the analytics leading to the Black and Scholes partial differential equation and then fully stop. At this point various specialized approaches and option valuations are presented to the students who are then told that if they differentiate diligently for a long enough period the given valuation will, in fact, satisfy the Black and Scholes equation. As a result, the students gain only marginal exposure to hedging and synthesizing various risk profiles and the art of financial engineering/risk management is relegated to a very special breed, the rocket scientist.

In this article, we introduce a simple mathematical scheme that addresses both issues raised above. A set of derivative security building blocks is identified. These atomistic options can then be combined to form a wide class of derivative structures and once in place the elemental options which we term "atoms" need only modest rebalancing. Furthermore, the technology is sufficiently straightforward that it can be easily introduced as a teaching tool to the advanced MBA. Let's now turn to a search for our derivative "atoms" and the related periodic table.

2. *Constant Risk Derivatives: The Atoms*

The non-diversifiable risk of a stock with respect to an index is often measured by the quantity beta. Similarly, the beta of a derivative security with respect to its underlying

instrument is termed the omega of the derivative[1]. In a single factor world, this measure of risk is simply the elasticity of the derivative security which we will identify as a stock for the purposes of discussion. Let's ask ourselves if there is a class of derivative securities with a <u>constant omega</u>. To confuse matters we label such a constant omega as n. Therefore, we search for a derivative security, $A_n(S,t)$, which satisfies both the Black and Scholes equation and the constant elasticity condition:

$$\frac{S}{A_n}\frac{\partial A_n}{\partial S} = n \quad \text{with } n \text{ constant} \tag{1}$$

Here $A_n(S,t)$ is the derivative value and S is the value of the associated stock. The solution to (1) is simply:

$$A_n(S,t) = K(t)S^n \tag{2}$$

where $K(t)$ is a function only of time.

Substitution of (2) back into the Black and Scholes equation[2] leads to an ordinary differential equation of order one, which can be readily solved, so that:

$$K(t) = K(0)e^{-\left[n(n-1)\frac{\sigma^2}{2}+(n-1)r\right]t} \tag{3}$$

Where σ and r are the volatility and the risk-free rate respectively. Combining factors and setting $K(0) = 1$, the derivative atoms, $A_n(S,t)$ are:

$$\boxed{A_n(S,t) = e^{-\left[n(n-1)\frac{\sigma^2}{2}+(n-1)r\right]t}S^n} \tag{4}$$

We are using the term "derivative atoms" interchangeably to mean either the function $A_n(S,t)$ or the derivative security $CA_n(S,t-T)$ with a payoff of CS_n at time T (C is a constant of calibration).

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[1] Of course, the Beta of a derivative security with respect to the market is simply Omega times the Beata of the underlying stock.

[2] For reference we restate the Black and Scholes equation: $\left(\frac{\partial f}{\partial t}\right)+rS\left(\frac{\partial f}{\partial S}\right)+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$

Although the equation (4) is true for any real number n, let's for the moment specialize to integral n's. Later we will relax this constraint. The first few atoms of the infinite sequence are set forth in the following table:

Table 1: Atoms	
n	$A_n(S,t)$
0	e^{rt}
1	S
2	$e^{-(\sigma^2+r)t}S^2$
3	$e^{-(3\sigma^2+2r)t}S^3$
•	•
•	•
•	•

Notice that the first two atoms of constant risk are simply <u>cash</u> and <u>stock</u>! Thus, the full "periodic table" of $A_n(S,t)$ is the natural extension of cash and stock, the normal participants in the hedging drama. Before moving on we note that the third atom $A_2(S,t)$ is a derivative of "almost" constant gamma. In particular:

$$\Gamma = \frac{\partial^2 A_2}{\partial S^2} = 2e^{-(\sigma^2+r)t} \tag{5}$$

For typical values of $\sigma = .3$ and $r = .1$, we have:

$$\Gamma = 2e^{-.19t} \tag{6}$$

which has a slow relaxation time of about five years; thus it takes about five years for Γ to fall to $\frac{1}{e}$ th of its original value.

In sum, we have found a set of atoms of constant risk that blend seamlessly into the existing dynamic hedging technology. The next issue to review is whether these atoms form a complete set. In other words can we combine these atoms in various proportions to build a wide class of derivative structures and once in place will rebalancing of the proportions be required?

3.   *Enter Weierstrass: A Complete Set*

Since $A_n(S,t)$ are solutions to the B-S equation, linear combinations of $A_n(S,t)$ will also

satisfy the classic PDE. Accordingly, for any selection of constant coefficients, a_n, the

following expression will also satisfy the Black and Scholes equation:

$$\sum_{n=0}^{m} a_n e^{-\left[n(n-1)\frac{\sigma^2}{2}+(n-1)r\right]t} S^n \tag{7}$$

The question, "Do we have enough atoms to build most derivative molecules" boils down

to the following. If $f(S,t)$ is the value of the derivative security, can we find a set of

constant coefficients, a_n, in expression (7) such that $f(S,t)$ can be approximately

arbitrarily closely and uniformly in a closed interval $[S_1,S_2]$? Fortunately, the answer is

yes.

Expression (7) is a power series in S and at first glance one would surmise that for large

enough m, the sequence of polynomials would effect and appropriate approximation of the

derivative payoff, $f(S,t)$, at expiration (time T). This surmise is confirmed in the

Weierstrass approximation theorem of 1885. Because of its importance in the present

context, a statement of the theorem is set forth below.

<u>Weierstrass Theorem</u>: Let $f(x)$ be a continuous function on the interval $[c,d]$. Given an

$\varepsilon > 0$, we can find a polynomial $P_m(x)$ (of sufficiently high degree m) for which

$$\left| f(x) - P_m(x) \right| \leq \varepsilon \qquad c \leq x \leq d$$

Thus the a_n's exist and the approximation of $f(S,T)$, with T being the expiration time,

by expression (7) works with the sole condition that $f(S,T)$ be a continuous function

over the closed interval $[c,d]$. Later, even the continuity condition will be relaxed.

In sum, carefully selected combinations of the derivative atoms will approximate as closely as desired any continuous derivative payoff while concurrently being a solution of the Black and Scholes equation. This also means that once these combinations (the a_n's) are determined the atoms need not be rebalanced over the lifetime of the derivative security. Since the a_n's exists, the next step is to find them. The process of extracting the a_n's we term spectroscopy since a spatial frequency composition is being determined. Before moving to the full spectroscope design, let's explore a few simple examples with exact solutions to extract the major themes.

4. *Derivative Atoms as a Teaching Tool: Some Simple Examples*

Two examples will suffice to give a flavor of the approach. First, let's begin with valuing a forward contract, $f(S,t)$, with a payoff at expiration (time T) of:

$$f(S,T) = S - K \tag{8}$$

From our earlier discussion we attempt to reconstruct $f(S,t)$ as:

$$f(S,t) = \sum_n a_n e^{-\left[n(n-1)\frac{\sigma^2}{2} + (n-1)r\right]t} S^n \tag{9}$$

At expiration, the contract value (see table of atoms) is:

$$f(S,T) = S - K = a_0 e^{rt} + a_1 S + a_2 e^{-\left(\sigma^2 + r\right)T} S^2 + \dots \tag{10}$$

For equation (10) to hold for all S, the coefficients must satisfy:

$$\begin{aligned} a_0 e^{rt} &= -K \\ a_1 &= 1 \end{aligned} \tag{11}$$

and all other a_n's are zero. Substitution of these coefficients into equation (9) yields:

$$f(S,t) = S - Ke^{-r(T-t)} \tag{12}$$

which is the standard formula for a forward contract on a stock with no dividend.

Similarly, let's quickly value a derivative security with a parabolic payoff of:

$$f(S,T) = (S-K)^2 \tag{13}$$

As in the case of the forward contract, we can express the derivative payoff as:

$$S^2 - (2K)S + K^2 = a_0 e^{rt} + a_1 S + a_2 e^{-(\sigma^2+r)T} S^2 + \ldots \tag{14}$$

Equating coefficients of S^n on both sides of equation (14) and substituting a_0, a_1, and a_2 into our basic equation (9), the resulting valuation expression for this derivative is:

$$f(S,t) = K^2 e^{-r(T-t)} - 2KS + e^{(\sigma^2+r)(T-t)} S^2 \tag{15}$$

Although illustrative, these examples mask a richer technology required in extracting the $a_n's$ in more general payoff profiles. Accordingly, the design of the spectroscope is now addressed, after which it will be used to examine a European Call.

5. *The Spectroscope: Basic Design*

The task at hand is to find coefficients b_n of the polynomial $P_m(S) = \sum_{n=0}^{m} b_n S^n$ such that if is as close as desired to a continuous payoff function $f(S,T)$ on the interval $[c,d]$. Then the coefficients in equation (9) will be given by:

$$a_n = e^{\left[n(n-1)\frac{\sigma^2}{2}+(n-1)r\right]T} b_n \tag{16}$$

To proceed with the analytic extraction of the coefficients (b_n's and $a_n's$), we define a measure of "closeness" which is fraternal twin of that used in the Weierstrass Theorem. Namely, define:

$$\|f(S,T)\| \equiv \int_a^b \left[f(S,T)\right]^2 dS \tag{17}$$

Then the extraction problem may be restated as finding an mth degree polynomial, $P_m(S)$ following the prescription:

$$\min_{b_0,\ldots,b_m} \|f(S,T) - P_m(S)\|^2 \tag{18}$$

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Here the function $f(S,T)$ and the polynomial of degree m are fixed and the minimization is performed over the coefficients b_n. The problem has a unique solution, $P_m^*(S)$, and as m gets larger the approximation improves. Also, in this modified measure of closeness, we can relax the requirement of continuity on $f(S,T)$ do that it only be sufficiently integrable.

This problem has a simple but elegant classical solution in the form of Legendre Polynomials. If we let $q_n(S)$ be the normalized orthogonal Legendre Polynomial of degree n on $[c,d]$, then $P_m^*(S)$ is the polynomial[3].

$$\boxed{P_m^*(S) = \sum_{n=0}^{m}\left(\int_a^b f(S,T)q_n(S)dS\right)q_n(S)} \tag{19}$$

and it is subsequently an easy matter to rewrite tis expression I terms of powers of S which will do in the example below. The series (19), called the truncated Fourier expansion for $f(S,T)$ converges uniformly on $[c,d]$ in the limit as $m\to\infty$.

We now illustrate those computational procedures with the example of a European Call.

6. *Illustration: The European Call*

Let's walk through a simple calculation to arrive at the atomic structure of a European Call with a payoff at expiration of $f(S,T) = Max(S-100,0)$.

Following the prescription set forth in equation (1), we list for reference the first few normalized Legendre Polynomials, $\bar{q}_n(S)$, which are orthonormal on $[-1,1]$.

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[3] See for example, Philip J. Davis, *Interpolations and Approximation*, Blaisdell Publishing Company, New York, 1965.

n	$\bar{q}_n(S)$
0	$\sqrt{\frac{1}{2}}$
1	$\sqrt{\frac{3}{2}}\,S$
2	$\frac{1}{2}\sqrt{\frac{5}{2}}\left(3S^2-1\right)$
3	$\frac{1}{2}\sqrt{\frac{7}{2}}\left(5S^3-3S\right)$
4	$\frac{1}{8}\sqrt{\frac{9}{2}}\left(35S^4-30S^2+3\right)$

To convert to orthonormal polynomial, $q_n(S)$, on $[c,d]$, use the transformation:

$$q_n(S)=\bar{q}_n\left(\frac{2(S-c)}{d-c}-1\right)$$

In this example $c=0$ and $d=200$. Then equation (19) asks for the computation of $\int_0^{200} Max(S-100,0)q_n(S)\,dS$ which is straightforward. Accordingly, the first five polynomials $P_m^*(S)$ approximating the payoff $f(S,T)$ are[4]:

|	$P_n^*(S)$: Legendre Form
n	
0	$35.36q_0(S)$
1	$35.36q_0(S)+40.82q_1(S)$
2	$35.36q_0(S)+40.82q_1(S)+19.76q_2(S)$
3	$35.36q_0(S)+40.82q_1(S)+19.76q_2(S)+0q_3(S)$
4	$35.36q_0(S)+40.82q_1(S)+19.76q_2(S)+0q_3(S)-4.419q_4(S)$

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[4] Note: All coefficients are rounded to four significant places.
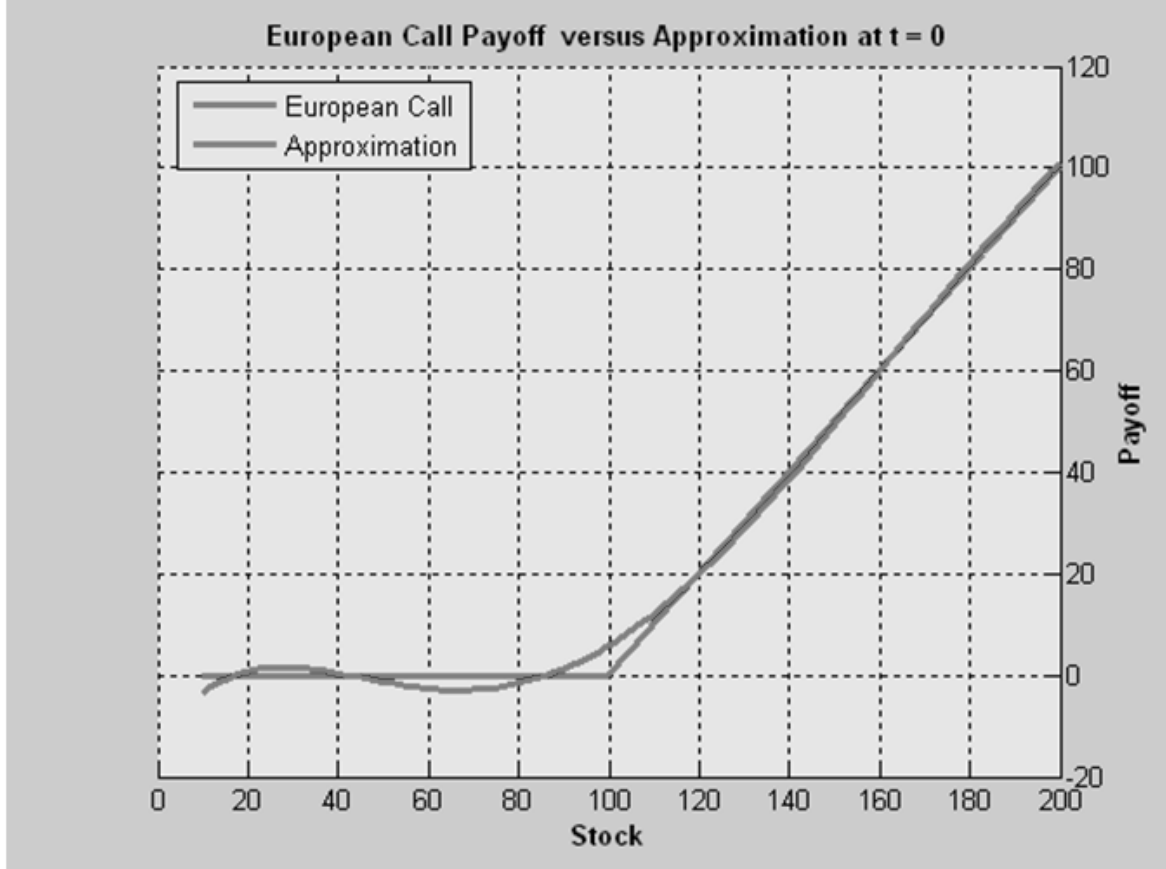
$$P_n^*(S): \text{ Unraveled Form}$$

n	
0	25.00
1	$-25.00 + 50.00(S/100)$
2	$6.250 - 43.75(S/100) + 46.87(S/100)^2$
3	$6.250 - 43.75(S/100) + 46.87(S/100)^2 + 0(S/100)^3$
4	$-3.125 + 50.00(S/100) - 164.1(S/100)^2 + 164.1(S/100)^3 - 41.02(S/100)^4$

A graphical comparison of the payoff profile with $P_2^*(S)$ and $P_4^*(S)$ are displayed below:

$$P_2^*(S) \text{ and European Call Payoff}$$

$P_4^*(S)$ and European Call Payoff

European Call Payoff versus Approximation at t = 0
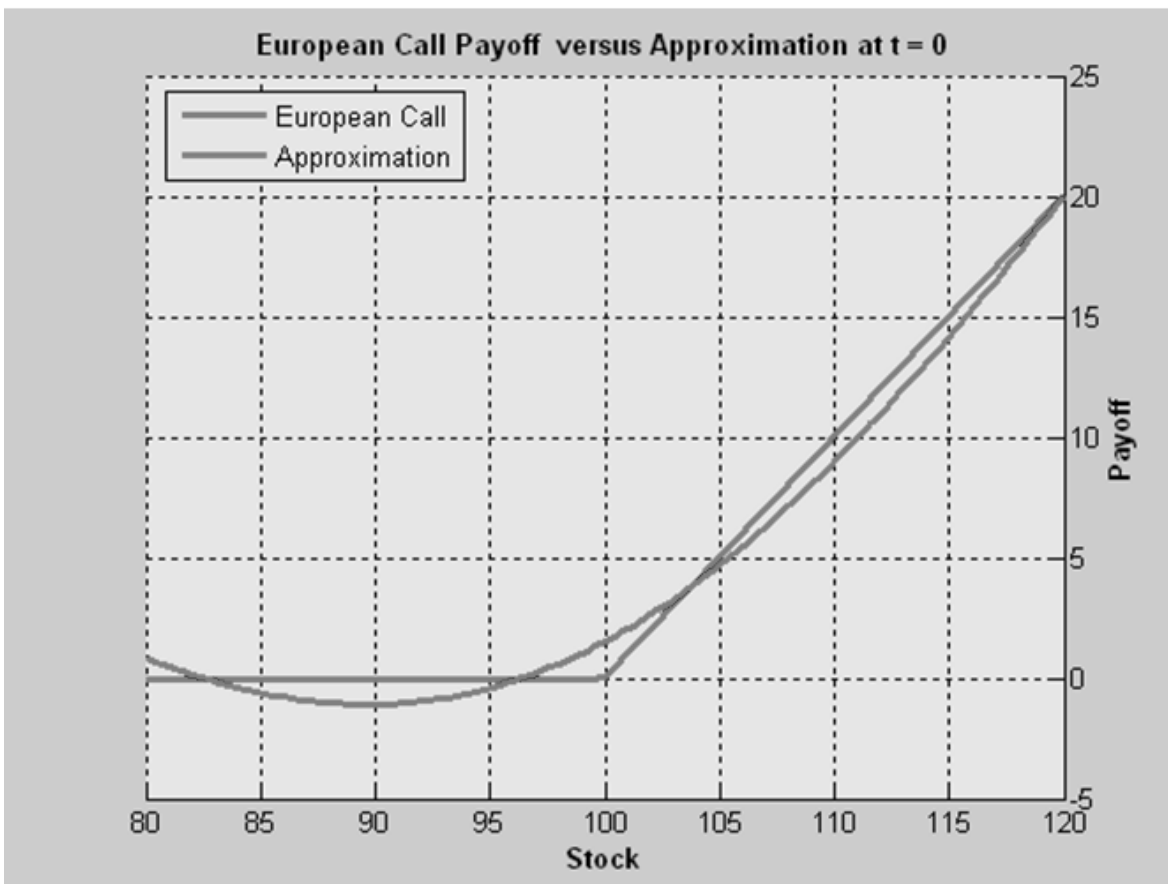
The rapid convergence to $f(S,T)$ with only a fourth degree polynomial is clear and relatedly only three atoms in addition to the traditional cash and stock are needed to mimic $f(S,T)$ to the degree of closeness specified by $P_4^*(S)$.

The unraveled form of $P_n^*(S)$ may then be used directly (see equation 9) to obtain approximations for $f(S,t)$. For example, for the fourth degree approximation with $0 \le S \le 200$ and $0 \le t \le T$:

$$f(S,t) \cong -3.125e^{r(t-T)} + 50.00(S/100) - 1.64.1e^{(\sigma^2+r)(T-t)}(S/100)^2$$
$$+164.1e^{(3\sigma^2+2r)(T-t)}(S/100)^3 - 41.02e^{(6\sigma^2+3r)(T-t)}(S/100)^4$$

(20)

and the five derivative atoms needed to synthesize $f(S,T)$ reflect within a calibration constant, the separate terms in equation (9). Again, the foregoing procedure may be

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viewed as a solution technique for derivatives as well as a motivation for the market to trade derivative atoms with payoffs proportional to S^n. A few such atoms can mimic the valuation of dozens of vanilla and exotic options of a given underlying security to a degree where investors would surely be indifferent between the original derivative and atomic cocktail. As demonstrated, the spectroscope fundamentals are transparent, and it is an easy task to design appropriate user friendly software for the graduate student or practitioner to extract the a's (the number of each type of atom to form a more complex derivative molecule).

Having covered the spectroscope basics, let's now briefly look at some extensions of the theme.

7.   *Spectroscopic Extensions*

The present approach to the design of the spectroscope allows for a number of extensions and generalizations. First, the function $f(S,T)$ need not be continuous but merely sufficiently integrable[5] so that one can even approximate Heaviside functions using this technology.

Secondly, the approximation of $f(S,T)$ can be extended from polynomials to functions of the form:

$$\sum a_n x^{c_n}$$

where c_n are simply real numbers. A powerful theorem due to Müntz[6] provides for functions $f(S,T)$ which are in L^2. It only requires that c_n are all distinct within a finite interval, say $c_n \in [1,2]$, to obtain appropriate convergence to $f(S,T)$. As an illustration, the minimization problem articulated in equation (18) can be explicitly solved by also allowing the powers of S as well as the coefficients to be variables. For the case of the

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[5] Specifically, $f(S,T)$ must be measurable and $\int_a^b |f(S,T)|^p dS$ must exist in which case $f \in L^p[c,d]$ for some $1 \le p \le \infty$. That is the case for functions with mild discontinuities.

[6] See Davis, page 272, for a complete statement of the Müntz theorem.

European Call described earlier, the approximation with only two terms (modified atoms) is[7]

$$Max(S-100,0) \simeq -3,984.38\left(\frac{S}{100}\right)^{1.8155} + 3,993.83\left(\frac{S}{100}\right)^{1.8231}$$

Note that $c_n \in [1,2]$.

Solving (18) as a minimization problem also allows us great flexibility in adding constraints. We could require the graph of the approximation to pass through the origin; the point $S=100$, $f=0$, the points $S=200$, $f=100$, or some other combination of these. We can also add restrictions on the shape of the approximation such as monotonicity for example.

One last extension is particularly noteworthy. The rationale for a constrained minimization often requires the approximation to pass through points where the payoff is highly likely to occur. This objective may be translated in general as follows: If the probability distribution of likely outcomes of f on $[c,d]$ is $w(S)$, then one would like the approximations to be closer to f where the probability of such outcomes is high. Then the natural generalization of the norm of f (equation (17)) is:

$$\|f(S)\|_w^2 = \int_c^d f^2(S)w(S)dS \tag{21}$$

where $w(S) \geq 0 \quad \forall S \in [c,d]$ and $\int_c^d w(S)dS > 0$. Recall that in the earlier formulation equal weights are given to each point in the approximation. Then all of the prior results can be obtained either by generating orthogonal polynomials[8] or directly through minimization. The topics are discussed in more detail in our forthcoming paper.

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[7] This approximation technique with two kinds of atoms is as good as the previous technique with four atoms. In our forthcoming paper we discuss various different kinds of atoms.

[8] Certain functions give rise to well-known classical polynomials. For example, $w(S) = (1-S^2)^{-\frac{1}{2}}$ give rise to the Chebyshev polynomials.

8. *Multifactor Considerations*

The extension of the foregoing schema to a multi-factor world is straightforward and only the essential elements will be outlined here using the vehicle of a two-factor derivative security. Let the factors, S_1 and S_2, be standard Brownian processes, so that:

$$\frac{dS_1}{S_1} = \mu_1 dt + \sigma_1 dz_1$$
$$\frac{dS_2}{S_2} = \mu_2 dt + \sigma_2 dz_2$$

(22)

and call the value of the derivative security $f(S_1, S_2, t)$. The two-factor PDE for f is then:

$$\frac{\partial f}{\partial t} + rS_1 \frac{\partial F}{\partial S_1} + rS_2 \frac{\partial f}{\partial S_2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} + \sigma_1 \sigma_2 \rho S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} = rf \qquad (23)$$

where ρ is the correlation coefficient between the natural logarithms of the returns on S_1 and S_2. It is natural to search for two-factor derivative atoms of the form:

$$A_{mn}(t) = K_{mn}(t) S_1^m S_2^n \qquad (24)$$

Following the procedure fully analogous to the one-factor methodology, the expression for $A_{m,n}(t)$ is introduced into the two-factor Black and Scholes equation (equation (23)). The resulting first order differential equation can be readily solved for $K_{mn}(t)$, again calling $K_{mn}(0) = 1$, the two-factor derivative atoms are: so that the two-factor atoms are:

$$A_{mn}(S_1, S_2, t) = e^{-\left[\frac{1}{2}m(m-1)\sigma_1^2 + \frac{1}{2}n(n-1)\sigma_2^2 + \sigma_1 \sigma_2 \rho mn + (n+m-1)r\right]t} S_1^m S_2^n \qquad (25)$$

Restricting for now (m,n) to integer pairs, the first few atoms may be depicted as:

(m,n)	A_{mn}
$(0, 0)$	e^{rt}
$(1, 0)$	S_1
$(0, 1)$	S_2
$(1, 1)$	$e^{-(\sigma_1\sigma_2\rho+r)t}\,S_1 S_2$
$(2, 0)$	$e^{-(\sigma_1^2+r)t}\,S_1^2$
$(0, 2)$	$e^{-(\sigma_2^2+r)t}\,S_2^2$
$\vdots$	$\vdots$

Finally, the general solutions of equation (23), $f(S_1 S_2, t)$ on a closed intervals $c \leq S_1 \leq d$ and $g \leq S_2 \leq h$ can be represented as:

$$f(S_1, S_2, t) = \sum_m \sum_n a_{m,n} A_{m,n}(S_1, S_2, t) \tag{26}$$

with $a_{m,n}$ being constant. The major conceptual features are quite parallel to the one-factor case. The multifactor spectroscope design, which is described in our forthcoming paper, also closely tracks its one-factor counterpart.

An interesting application of the two-factor approach (with a few variations on the theme) is to revisit Robert Merton's valuation of options incorporating stochastic interest rates[9]. Merton's basic PDE s a special case of equation (23) with r set to zero, identification of S_2 with the discount bond P and allowing σ_1, σ_2, ρ to be time dependent. The well-known zero investment criteria for the hedge portfolio is:

$$f = S\frac{\partial f}{\partial S} + P\frac{\partial f}{\partial P} \tag{27}$$

[9] Merton, R. C. 1973. "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science*, 4, Spring, 141-183.

Applying this condition to $f(S_1, S_2, t)$ in equation (23) or equivalently to $A_{m,n}(t)$ in equation (25) uncovers the coupling of m and n: $m+n=1$. This simple relationship is the basis of normalization procedures introduced by Merton in the mid 70's. Namely, substitution of this m,n connection into equation (25) reveals:

$$f(S_1, S_2, t) = \sum_m a_m K_{m,1-m}(t) S^m P^{-m+1}$$

$$\frac{f}{P} = \sum_m a_m K_{m,1-m}(t) \left(\frac{S}{P}\right)^m$$

(28)

Thus, the two-factor atoms provide a user-friendly introduction to the well-known homogeneity property depicted above. Further, requiring that $f(S, P, t)$ satisfy Merton's PDE allows for a one-step determination of $K_{m,m-1}(t)$ with the results that $\frac{f}{P}$ (equation (28) may now be specified as:

$$\frac{f}{P} = \sum_m a_m e^{-m(m-1)\frac{\bar{V}^2}{2}t} \left(\frac{S}{P}\right)^m$$

$$\text{where } \bar{V}^2 t \equiv \int_0^t \left(\sigma_1^2(s) - 2\sigma_1(s)\sigma_2(s)\rho(s) + \sigma_2^2(s)\right) ds$$

(29)

It is transparent from the above that this solution can be mapped into the one-factor case (see, for example, equation (9)) with $P \to e^{rt}$, $\frac{S}{P} \to S$, $\bar{V}^2 \to \sigma^2$, as Merton demonstrated over 40 years ago.

9.   *Summary and Market Challenge*

In this article, elementary derivative functions (atoms) are identified that satisfy the Black and Scholes equation and which, in aggregate, can be combined to form a wide class of derivative security risk profiles. This solution technique for valuing derivatives has the potential for serving dual roles. As a teaching tool, the analytics are readily accessible at the intermediate graduate level of study and could be a useful sequel or chapter to standard texts introducing the Black and Scholes PDE. If, on the other hand. the market elected to trade even a handful of these "atoms" for a given underling security or index, both

speculators and hedgers could benefit. Speculators could carefully select their risk class with a derivative instrument bearing a constant elasticity relative to the underlying throughout the life of the contract. In turn the hedger would find a user-friendly technology based on Legendre techniques to assemble combinations of the derivative atoms to form more complex derivative risk profile (molecules), once in place, the synthesized structures require only limited rebalancing. In sum, the challenge to the market is simply this … *issue these derivative atoms* and provide the building blocks of a true financial "engineering of things."