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Strategic foreign reserves risk management: Analytical framework

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Abstract We present an analytical framework for active foreign exchange reserves management that integrates risk-return objectives with macroeconomic, macro-prudential and sovereign debt management concerns. Our framework allows for very general objective functions, does not restrict the class of eligible stochastic processes or limit the investment universe, and can incorporate many types of macroeconomic concerns. It incorporates several kinds of risk constraints in order to obtain benchmarks satisfying possible central bank requirements of safety, liquidity, returns, and stability. Feedback between outcomes and decisions is easy using tools that reshape distributions and functions of the outcomes. And the model can be run on a PC-based platform. We apply the framework to several common reserves management problems focusing especially on the formulation of model equations, generation of trees and estimation of density functions of outcomes. We compare our approach to those used by many central banks and discuss advantages to our approach.

Keywords Reserves management · ALM · Asset/liability · Dynamic stochastic optimization

Central bank foreign exchange reserves risk management concerns balancing many objectives and issues, from broad macro-economy policy objectives, such as monetary policy and foreign exchange management, to micro-aspects, such as the definition of portfolio benchmarks and the evaluation of investment managers. Furthermore, constraints arising from legal, human resources, asset markets, institutional and other aspects affect the actual achievability and implementation of reserves management objectives. While the macro-economic aspects

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of reserves management have been much analyzed, they continue to draw attention, especially the relationship between financial crises and reserves and debt management. Many crises in the last decade have underscored relationships between foreign exchange reserves and public debt management and highlighted how private debt management can exacerbate economic or financial shocks. The micro-aspects of improving reserves management have also received much more attention with the wider investment universe and the broader range of financial tools that have become available. These developments have led to more active reserve management strategies and methods applied by private asset managers to find their way into foreign exchange reserves management.

While these macro and micro issues have been considered for some time and are analyzed in much detail in various strands of literature, often these aspects are not being considered jointly, at least not within one, common analytical and empirical framework. The analysis of tactical investment management, for example, often takes the achievement of certain macro-objectives as given and focuses mainly on assessing more narrowly defined risk-return tradeoffs. And vice-versa, the macro-literature has often focused on the links between foreign exchange reserves management and exchange rate determination, but has given little consideration to the investment choices made, say in terms of currency and maturity.

Recent experiences suggest that these assumptions of separation cannot be taken for granted since there are many relationships and feedbacks between the various macro- and micro-economic aspects and between foreign exchange reserves and liability management. The financial crises literature, for example, has drawn attention to the need to consider the overall balance sheet of the government, and indeed of the country as a whole, when conducting foreign exchange reserves and debt management. Literature has also highlighted how private debt management and financial sector vulnerabilities can affect public debt and reserve management. The starting point to integrate the various, sometimes-disjoint approaches has to be an analysis of what a general framework should look like. One can then assess what is feasible computationally and organizationally. We will show in this paper that recent technological and analytical advances allow for very general approaches that at the same time can lead to outputs that are easily understandable and tools that are easily implementable as well as adjustable.

Our approach is based on a general, dynamic stochastic optimization model with a tree-based uncertainty structure. The numerical approach allows us to move beyond the classical methods of risk and return, efficient frontiers, and utility functions by considering much more general objective functions, behavioral relationships, equalities, and inequalities. The approach also innovates by allowing the user to view the full density functions of any outcomes that depend upon the optimal decisions taken. The user can then reshape these density functions as much as possible to any desired profiles and obtain the resultant decisions to be taken today to achieve these profiles. As we will show, this approach allows senior management to define relatively broad objectives and constraints, and have those translated into an analytically rigorous approach, without giving up basic intuition and understanding.

The outline of the paper is as follows. We first review the typical approaches used for foreign exchange reserves risk management. We then present our general approach to asset/liability management (ALM), including reserves management, using a dynamic stochastic optimization model. This is followed by several applications of our approach to the problem of foreign exchange reserves management by a central bank, along with some specific examples. Throughout the paper we use the word "ALM" in the most general sense, that is, to refer to the analysis and management of risks related to any combination of assets, liabilities, off-balance sheet items, and any cash flows that could affect the market value of the portfolio or the objective function, including contingent liabilities and catastrophic events. We do not restrict ALM to mean asset-liability matching; rather our analytical framework allows a much richer universe of options, constraints, and objectives in the analysis.

1 Typical approaches to foreign exchange reserves risk management

Approaches to reserves management vary along a wide spectrum. At opposite ends of the spectrum are the traditional, pure macro-economic-only oriented approaches and the pure micro-based, risk-return-only oriented approaches. We discuss the macro-economic approaches first. The traditional reserves management objectives have mostly been formulated with respect to monetary policy and exchange rate management. These macro-economic concerns are most relevant in the context of fixed or managed exchange rate regimes with some degree of capital mobility. In that case, foreign exchange reserves needs to buffer against capital outflows in excess of the trade balance (and reserves will increase as capital flows exceed any trade deficit and vice-versa). Reserves management is then secondary to the macro-economic objectives and mainly involves liquidity management, which is assuring the availability of sufficient free reserves at each moment in time to intervene in the foreign exchange market.

More generally, under fixed as well as floating exchange rate regimes, the macro-economic literature views holding reserves as a way to smooth short-run shocks in external transactions, such as variations in imports due to terms of trade shocks or variations in the capital account due to financial shocks. Rules have been developed for these circumstances, along the lines of optimal cash inventory management for corporations. The simplest practical application of this approach has taken the form of targeting a minimum ratio of foreign exchange reserves to imports, e.g., holding foreign exchange reserves at least equivalent to 12 months of imports. These and other, more sophisticated rules for a country's demand for international reserves have mostly considered only real variables, such as imports, exports, and the severity of possible terms of trade shocks, as well as some monetary policy considerations (e.g., Frenkel (1980), Frenkel and Jovanovic (1981), see Bahmani-Oskooee and Brown (2002) for a recent review). Most of this research, however, does not consider financial or balance sheet variables.

In the last decade, the relationship between (external) sovereign debt and reserves management has attracted much interest. Besides the total level of (external) sovereign debt, the importance of the maturity structure of debt relative to available reserves has been much highlighted. Governments typically find it relatively inexpensive to borrow short term, since spreads usually contain a term premium, especially so for emerging markets' governments. But a high proportion of short-term debt tends to increase the probability of self-fulfilling crises, as investors might suddenly decide not to roll over maturing debt or increase required yields on new debt. With foreign exchange reserves low relative to debt payments falling due, the risk of a currency or financial crisis can increase sharply. The structure of public debt, especially its maturity and currency composition, indeed has had key implications in many recent crises or near crises. In those cases, a significant share of public debt was issued in foreign currency and/or linked to foreign currency and contributed to rapid increases in debt ratios when exchange rates depreciated, inviting further pressure on reserves and the exchange rate as it generated doubts about the sustainability of public or private sector finances.

The literature that studies these relationships between government debt levels, maturity structure and reserves management and financial crises is rapidly growing and includes Calvo and Mendoza (1996), Sachs, Tornell, and Velasco (1996), Rodrik and Velasco (1999), and Jeanne (2000). The literature has led to the various rules on what are acceptable levels of debt payments falling due relative to the level of foreign exchange reserves (named after their principal or first advocates, the Calvo, Guidotti, and Greenspan rules). One rule of thumb has been that a country's government external debt repayments falling due in the next 12 months should not exceed its foreign exchange reserves. In some crisis countries, private asset and liability management has also contributed to pressures on reserves by not providing for sufficient buffers against shocks. Consequently, some rules have included private as well as public debt payments falling due relative to foreign exchange reserves. In general, the structure of balance sheets, whether of the sovereign, the banking system or the corporate sector have received more attention among reserve managers lately as reserves' key role as a buffer in relation to both public and private debt has become clearer.

Less macro-economic or balance-sheet oriented approaches and more micro-economic oriented approaches to reserves management are feasible when monetary policy, exchange rate and debt management issues are of less concern, and when vulnerabilities in the financial and corporate sectors are small. This may be the case when the government pursues a flexible exchange rate policy, when it has a credible fiscal policy and institutional framework, and when domestic financial markets are well developed. A more active approach to reserves management may then put greater emphasis on profit objectives, although still within certain bounds. The implementation of this objective may involve the division of the reserves portfolio into active and passive parts. The passive portfolio would be used to deal with macro-economic objectives, and be mainly managed with liquidity objectives in mind. The active portfolio could be used for profit-purposes, possibly taking into account liability management objectives. Its management would consider the breadth of tactical risk management tools available, including many types of investments and borrowings, and forwards, swaps, plain vanilla and exotic options, etc., and use concepts such as value-at-risk.

The division into two portfolios may, explicitly or implicitly, involve the separation of objectives, as in the case of separate management where it may be assumed that there will be little or no spillover between the two parts, at least over the horizons set for the managers. It will typically also involve different institutional arrangements for the management of the two portfolios as well as different investment tools or instruments to be used. In terms of ALM, the management of the passive portfolio could be similar to the general, more macro-oriented liquidity management raised above. The ALM of the active portfolio would appear to come very close to the ALM-issues facing a commercial bank or a corporation. While correct in many ways, there still remain considerable differences, however, such as the difficulty by which measure to judge the performance of the portfolio (e.g., in which currency should the profits and risks of a central bank be measured, the local currency or one of the intervention currencies, e.g., US dollar, Euro or Yen?). Importantly, the (implicit) assumption of separation would need to be reviewed regularly.

In practice, these macro and micro-based objectives and accompanying tools for reserves management will often be mixed or take a hierarchical form. Typically, approaches used for central bank reserves management involve combinations of many objectives, with four main concerns often mentioned, usually in this order: security, liquidity, profit, and stability. Correspondingly, analytical ALM-models may try to define one or more of these objectives. For example, the central bank may make its ALM-objective trying to achieve those asset and currency allocations for which the probability of negative unrealized losses is very small. Or the objective may be to have sufficient liquidity available for intervention in 99% of cases. Or the objective may be that the rates of return need to be between 3% and 4%, while at the same time the ratio of short-term-debt to reserves should not exceed 1.05 for 99% of the time. Or the objectives might be stated as some combination of these or others.

In the next two sections we will describe our analytical framework and its application to specific reserves management problems that will provide strategic allocations of assets, currencies, and liabilities to meet these kinds of mixed macro and micro-based objectives. The analytical framework is a dynamic stochastic optimization model.[1] Whereas in the past these kinds of models may have taken a long time to build and deploy, with the confluence of several new developments in mathematics, computers, algorithms, modeling languages, and tools for insight and intuition, this is no longer the case. This paper is based upon a research project to develop a conceptual framework for sovereign ALM undertaken at the World Bank during the 90's and summarized in Claessens et al. (1998). An application of this framework to the Republic of Colombia is provided in Claessens et al. (2000). The analytical framework also has a corresponding computational framework that makes it possible to formulate, solve, and implement these models very rapidly, which is discussed in Kreuser (2002). This is very important because, even though we will specify precisely the equations for a prototype model, each central bank or government will have its own unique modeling requirements. The idea is that it is easy to modify these models to meet those specific requirements.[2]

In Section 4 we will discuss creating the framework for a specific problem for a central bank. We turn to the IMF (2003) paper to discuss individual cases of central banks' current practices for managing reserves and to illustrate the types of countries that would benefit from the specific applications in this paper.

2 A general analytical framework for ALM

The approaches currently taken by central banks (and governments) show that it is unlikely that a single reserves management model will be applicable in all situations. No single model can be expected to be optimal for every central bank's reserves management requirements. Furthermore, every model developed today will need to change tomorrow. Flexibility must thus be built into any approach to reserves management modeling. The purpose therefore should be to provide a framework that allows for substantial flexibility in model development in both theory and application. A framework has to be dynamic as well. Any medium to long-term analysis of reserves must include procedures for the dynamic rebalancing of the reserves portfolio for two reasons: the density functions of outcomes in the far future depend upon decisions taken beforehand, that is in the near future; and changing regime conditions mean future decisions need to be made dependent on future outcomes.

The need for strategic ALM for sovereigns is all the more necessary, as sovereigns often have to consider risks on a much broader scale than corporations or financial institutions do. Risks for a sovereign concern not only the government's own direct financial exposures, such as those arising from debt and reserves management, but also those arising from contingent liabilities due to risks in the banking system, restructuring of state-owned enterprises, or restructuring and reform of the corporate sector. Approaches also need to relate to measures of the government's earning potential. This may mean risks need to be defined differently. Instead of measuring nominal variability in government debt service, for example, risk measures may need to take into account the sensitivity of fiscal revenues to global factors, such as interest rates. Without these factors, approaches to risk can ignore the existence of natural hedges in the external and fiscal sectors, limit the analysis to "on-balance" liabilities only,

[1] Birge and Louveaux (1997) discuss dynamic stochastic optimization, or stochastic programming, in general and include some financial examples. Examples of dynamic stochastic optimization models with objectives, equalities, and inequalities can be found in Ziemba and Mulvey (1998).

[2] For those less interested in technical details, Claessens and Kreuser (2004) provide an overview of the approach discussed in this paper in the book published by the European Central Bank.

and ignore many important constraints. It is also essential that the sovereign adopts a truly dynamic approach. Countries often face many constraints in rapidly adjusting their assets and liabilities; transactions costs can be high and market access may vary with general financial market conditions and investors' sentiment, especially for developing countries but also for developed countries. Typical ALM strategies pursued for financial institutions and corporations can therefore be less than optimal for sovereigns or central banks and measuring exposures only in terms of duration, asset composition, and currency composition may even add to risk or costs.

Our analytical framework is based on three main pillars:

(1) The generation of sparse trees of stochastic variables;
(2) The formulation and solution of a dynamic stochastic optimization model; and
(3) The shaping and estimation of density functions of outcomes from the model.

The idea is that the asset and liability allocations today will be determined on the basis of visualizing the density functions of outcomes that are shaped using various combinations of techniques.

2.1 Trees of stochastic variables

In any ALM model, the variables that define the future states of the world are of two classes. The first are the exogenous variables that may include interest rates, exchange rates, current account deficits, and others that one wishes to stochastically estimate. The second are the endogenous or "decision variables," that is, those determined by the model. They could include the amount of reserves and debt, the maturity structure of foreign exchange reserves and debt, the percent of each reserve currency, etc. The initial step in any analytical framework with regard to the exogenous variables is to generate a collection of scenarios of possible realizations. These exogenous variables chosen could include interest rates, exchange rates, bond prices, GDP, trade deficit, fiscal deficit, commodity prices, as well as those rarer events triggering contingent liabilities (e.g., a banking system crisis) and other factors.

There are two distinct topologies that are usually considered for defining the future states of the world: a scenario structure that is obtained by generating states of the world along sample paths or scenarios that have no branching structure; and a tree structure, that is, generating trees so that future states have events with future states branching out of them.

There are several reasons why a tree structure is better than a scenario structure. First, the tree branching structure provides a good discrete approximation of the stochastics of most processes, allows many interdependent factors to be generated, can be adjusted easily to satisfy the stochastic properties of processes, and is a natural representation of the way uncertainty unfolds. This structure is necessary for optimization models for the reasonable calculation of future distributions and densities of endogenous variables when portfolio rebalancing is present.

Future exogenous states of the world in our framework must be generated in such a way that it is possible to adjust them to also satisfy a combination of the following:

(1) Price and rate stochastics match historical information;
(2) Values (means, correlations, etc.) can be over-ridden by expert opinion;
(3) Prices and rates and their means and volatilities can be adjusted so that they match those implied by current derivative values; and
(4) Prices and rates can be adjusted to satisfy those conditions implied by various theories such as the Uncovered Interest Parity hypothesis.

The reason for these requirements is that they may be necessary for modeling a suitable base case and to undertake stress tests for the kinds of issues sovereigns encounter.

What is critical in our approach is that the tree topology does not become excessively large. There are several ways to control size. One way is to generate scenarios and then to apply scenario reduction techniques such as those found in Heitsch and Römisch (2003).[3] It is also possible to go from generating scenarios to bundling them to get trees. An example of this is given in Krokhmal and Uryasev (2003).

One way that we have found particularly useful for representing the behaviour of the exogenous variables and to generate scenarios is to use multi-factor stochastic partial differential equations such as:

$$\frac{ds_i(t)}{s_i(t)} = \mu_i(s,t)dt + \sum_j b_{ij}(s,t)\sigma_j(s,t)d\omega_j(t) \tag{1}$$

This representation[4] underlies commonly used multi-factor models, including those with mean-reversion, such as Hull-White and Heath, Jarrow, and Morton. The parameters (μ, b, σ) of these equations can be determined from historical data as discussed in Göing (1996).

We divide the interval of time from today, $t = t_0$ to the terminal period, $t = t_T$ into intervals $[t_0, t_1], [t_1, t_2], \ldots, [t_{T-1}, t_T]$ and use an estimate of μ, b, σ for each interval separately. This allows, for example, the correlations to depend on the time and the level of the variables, allowing the generation of a very rich possibility of outcomes. The solution[5] of these equations for each interval, in which μ, b, σ are constant is:

$$\begin{aligned}
s_i(t) &= s_i^0 \exp\left[\left(\mu_i - \frac{1}{2}\sum_{j=1}^n b_{ij}^2\sigma_j^2\right)t + \sum_{j=1}^n b_{ij}\sigma_j\omega_j(t)\right] \\
E\{s(t)\} &= \left(s_1^0 e^{\mu_1 t}, s_2^0 e^{\mu_2 t}, \ldots, s_n^0 e^{\mu_n t}\right) \\
\operatorname{cov}(s_l(t), s_k(t)) &= E\{s_l(t)\}\, E\{s_k(t)\}\left(\exp\left(t\sum_{j=1}^n b_{lj}b_{kj}\sigma_j^2\right) - 1\right) \\
\operatorname{var}(s_l(t)) &= (E\{s_l(t)\})^2\left(e^{t\sum_{j=1}^n b_{lj}^2\sigma_j^2} - 1\right)
\end{aligned} \tag{2}$$

These equations provide the first and second moments for solving the moment matching problem. A discussion of moment matching scenario generation is given in Høyland, Kaut, and Wallace (2003). We use a variation with variable probabilities because that allows the tree to be much smaller and yet still match the stochastics of (2). A tree structure with variable probabilities is important because it means that scenarios of low probability, yet with high impact can be added without requiring thousands of new scenarios to be generated. In addition, by building the tree event by event, one can trigger events to occur at certain

[3] Some of these techniques are also implemented in the General Algebraic Modeling System, GAMS as additionally provided modules. This makes it easy to go from theory to applying the techniques in a real model. Details and documentation can be found on http://www.gams.com.

[4] Because of the way we structure our model formulation, we estimate most variables in the rate-of-return space. This makes the formulation particularly consistent and applicable especially in the formulation of the stochastic estimations.

[5] Solution was provided by Roger J-B Wets while consulting on the World Bank project.

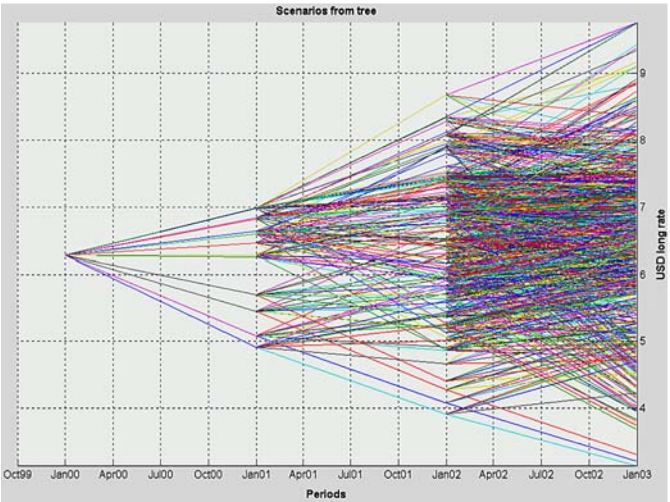

Fig. 1 Sample tree[6] defined over 3 periods

times or at certain levels of variables, such as low levels of GDP-growth triggering contingent liabilities in state-owned enterprises or guarantees on deposit insurance in the banking system falling due. This allows us to capture tail events and yet keep the tree relatively small. Using this variable probability process, trees of 40 to 50 correlated factors can be generated over several periods and solved on a PC.

An example of a tree generated in this manner is given in Fig. 1 and discussed in more detail in Kreuser (2002). This tree was generated as part of the study of optimal foreign exchange reserves management for the Republic of Colombia.

2.2 The structure of the dynamic stochastic programming models

The next step is to build a model to derive the decision variables. The model is defined independently of the stochastic processes, the tree, or the events on the tree. The separation of the stochastic processes from the model formulation makes the system very powerful. It becomes trivial to make changes to constraints or objectives or to modify the tree (e.g., make it larger or smaller or introduce stress tests) without affecting the tree or the model respectively.

In order to make the model formulation easy, we adopt the following notation to characterize the topology of the tree.

$$X^t \equiv \{e \mid \text{event } e \text{ occurs in time period } t\} \tag{3}$$

$$AT^{t,e} \equiv \{(\tau, \varepsilon) \mid \text{For } \tau < t, \text{ event } \varepsilon \text{ preceeds event } e \text{ at time } \tau\} \tag{4}$$

[6] The branches on the tree are not of equal probability. In order to visualize where the probability mass lies, a second graph is generated head on into the branches with the size of the branch or dot scaled depending on its probability mass. Alternatively, the density function or distribution is generated at each time period. See Claessens, Kreuser, and Wets (2000).

The set definition (3) indicates what events occur for each time period and (4) indicates the events that precede any event in any time period.[7]

Decision variables are then defined for each event with respect to the levels of assets, liabilities, currencies, alternative investments (derivatives), and other cash flows. Decision variables (including shorting) are all handled separately, allowing the specification of transaction costs, spreads, and limits to be individually imposed and thus increasing the stability and realism of the model. The selection of what decision variables to use depends on the specific issues to be analyzed and objectives pursued. Since the model allows analysis at the strategic level, assets in the portfolio are usually defined in terms of broad classes, such as short, medium, and long duration assets, or buckets of different currency classes, rather than at the level of individual assets. We might, for example, define an asset class as those short duration assets of no more than 6 months in a particular currency and make the percentage of the portfolio holdings of those assets the decision variable.

The sets that we will use to characterize the asset and liability classes, the currencies, and time periods are:

$$\begin{aligned} I &\equiv \{\text{set of all asset classes, } i\} \\ J &\equiv \{\text{set of all liability classes, } j\} \\ C &\equiv \{\text{set of all currencies, } c\} \\ T &\equiv \{\text{set of all time periods, } t \text{ with } t = \bar{t} \text{ the last time period}\} \end{aligned} \tag{5}$$

We define the variable[8] $A_{i,c}^{t,e,\tau,\varepsilon}$ as the holdings of asset class i in currency c purchased at time t and event e and held at time τ and event ε. We define the variable $B_{j,c}^{\tau,t,e}$ as the liabilities of class j in currency c purchased at time τ and held at time t and event e. The reason for the difference in indexing will become clear in Section 3.2. The details for the accommodation of assets or liabilities maturing between periods and for periods of variable length are left out for the sake of clarity.

We list the notation for the variables and parameters used in the model in the following tables. We use the convention of capital letters for those variables whose values are determined by the dynamic stochastic programming model, with the exception that capitals are also used for set names. Since we show two types of models, the column "Model" specifies whether or not the variable is used primarily in the reserves model, CB, or used primarily in the debt model, MoF, or for both.

[7] In the generation of the model each of the sets are actually produced. This mathematical notation can be directly translated into GAMS notation.

[8] We find this notation convenient especially for asset classes most often held by a central bank. The notation is different than is normally used as we measure the purchase value while others often measure the number or volume of assets. There are reasons to use either notation for specific problems. A notation similar to this was proposed by Lane and Hutchinson (1980) but it was first implemented commercially by Lane and Kreuser (1980). The Lane and Kreuser model was one of the first commercially applied dynamic stochastic programming models under uncertainty. It was used weekly in 1976–77 in strategy meetings for determining tactical portfolio movements of the World Bank's six billion dollar treasury portfolio.

Model variables	**Model**	**Definition**
$ALPHA^t_{ret}$ and $ALPHA^t_{liq}$	CB	Alpha variable in definition of CVAR constraints for returns and liquidity respectively.
$A^{t,e,\tau,\varepsilon}_{i,c}$	CB	Holdings at par at time τ and event ε of asset i in currency c acquired at time t and event e.
$B^{\tau,t,e}_{j,c}$	MoF	Borrowings in instrument j in currency c acquired in time τ and held in time t and event e.
$BL^{t,e}_{j,c}$	MoF	Original debt in instrument j in currency c held in time t and event e.
$CB^{t,e}_c$	CB MoF	Central bank excess profits and losses.
$LQ^{t,e}$	CB	Liquidity portfolio at time t and event e. Here it is assumed to be in US dollars.
PRF	CB MoF	Value of preference function.
$SG^{t,e}_1$, $SG^{t,e}_2$, $SG^{t,e}_3$	CB MoF	Variables for segmenting the piece-wise linear-quadratic form.
$TA^{t,e}$	CB	Total assets.
$TB^{t,e}$	MoF	Total debt.
$TRS^{t,e}$	MoF	Total transaction costs.
$V^{t,e}_{c,d}$	CB	Transfer of currency from c to d at time t and event e.
$ZLIQ^{t,e}$	CB	Shortfall in liquidity portfolio at time t and event e.
$ZRES^{t,e}$	MoF	Shortfall in ratio of reserves to short-term-debt.
$ZRET^{t,e}$	CB	Shortfall in returns in time t and event e.
$ZTRS^{t,e}$	MoF	Shortfall in transactions costs.

Tree variables	**Model**	**Definition**
$\alpha^{\tau,\varepsilon}_{j,c}$	MoF	Interest rate on debt j in currency c acquired in time τ and event ε.
$bas^{t,e}$	CB MoF	The exchange rate of the basket currency with respect to the numeraire at time t and event e. This is a function of the definition of the composition of the currency basket and the exchange rates $\gamma^{t,e}_c$
$cash^{t,e}_c$	CB	Cash in (+) or out (−) in currency c at time t and event e.
$\delta^{t,e}$	CB	Discount factor at time t and event e. Used in the preference function.
$exmx^{t,e}$	MoF	Maximum market activity for external debt.
$exports^{t,e}$	MoF	Exports.
$\gamma^{t,e}_c$	CB MoF	The exchange rate in terms of the amount of currency per numeraire at time t and event e.
$ldmx^{t,e}$	MoF	Maximum market activity for new debt in local currency.
$\eta^{\tau,t,e}_{i,c}$	CB	The price adjustment for asset i in currency c acquired at time τ and marked-to-market at time t and event e.

Tree variables	**Model**	**Definition**
$mx^{t,e}$	MoF	Maximum market activity in local currency.
$\phi^{t,e}$	MoF	Fiscal deficit.
$\pi^{t,e}$	CB MoF	The probability that event e will occur in time t.
$\rho_{i,c}^{\tau,\varepsilon}$	CB	Interest rate of asset i in currency c acquired at time τ and event ε. This assumes a fixed rate asset. The notation for variable rate assets is easy to define.
$std^{t,e}$	CB	Short-term debt at time t and event e.

Parameters/constants	**Model**	**Definition**
β_j^t	MoF	Principal payment percentage after t periods for debt j.
$cliq^t$	CB	Liquidity portfolio requirements in time t.
$cret_r^t$	CB	Is the compounded desired return on assets at time t at the rate r. For example, $cret_3^t = (1.03)^t \sum_{i\in I,c\in C} ina_{i,c}$.
$ina_{i,c}$	CB	The beginning portfolio in asset i and currency c.
$inl_{j,c}$	MoF	Initial debt.
lcb	MoF	Percent of excess reserves profits/losses transferable to/from debt portfolio.
$liqb$	CB	Initial liquidity portfolio.
ma_c^t	CB	Minimum activity policy constraint for currency c and time t.
mat_i	CB	Maturity of asset i.
mb_c	MoF	Minimum market activity in currency c.
$nret^t$	CB	Minimum return (realized or unrealized) on the portfolio for time t. This is a policy variable and also used for shaping the density function of the variability of returns.
$[p^t, q^t]$	CB MoF	Interval for each time period t in which one wants to push the probability mass.
pnb	MoF	Percent penalty payment for prepayment of debt.
r_1^t, r_2^t	CB MoF	The beginning and ending slopes of the linear parts of the piecewise linear-quadratic at time t. The values are set to push more or less probability mass into the interval $[p^t, q^t]$ for each time t. Typically we use $[r_1, r_2] = [100, .01]$ as a starting interval for maximization.
rcb^t	MoF	Return defining excess profits/loss of reserves portfolio.
tc_c	CB	Transaction cost for currency transfers in currency c.
tca_i	CB	Transaction cost of purchase of asset i.
tcl_j	MoF	Transaction costs on debt j.
tcs_i	CB	Transaction cost of sale of asset i.

The model constraints that might be imposed can be many and varied and examples are discussed in the next section. Some possibilities are: legal limits on asset classes; portfolio rollover constraints; transaction cost limits; cash flow requirements; currency transfer constraints; market access constraints; liquidity constraints; and other policy constraints. These conditions constrain the set of feasible strategies. Then there is a whole collection of constraints to shape the distributions of factors that are deemed important. We can call these objectives, although the distinction between constraints and objectives is not always clear. A simple example of an objective and a set of constraints that typically form the basis for many asset portfolio allocation problems is to maximize the expected return at the horizon with constraints governing the allowable purchase and sale of assets, including transaction costs. An example for a central bank's asset and currency allocation problem would be to state as objectives to achieve a high return, a low probability of unrealized negative returns and/or a low probability of portfolio sales, while maintaining a liquid portfolio to best meet intervention requirements. An example of objectives for the debt management of a ministry of finance would be to minimize the expected ratio of external debt to GDP, to limit the upside volatility of this ratio, to hedge against market liquidity, and/or to best meet new financing needs.[9]

Once the objective and constraints are defined, the model is solved simultaneously for all decision variables. The computational approach to solve these models is discussed in Claessens et al. (1998).

2.3 Estimating and shaping density functions

The technique for estimating a density function based upon a discrete distribution coming out of the model solution is given in Wets (1998). We solve the following maximum likelihood problem:

$$
\begin{aligned}
&\min - \sum_{e \in X^{\bar{t}}} \ln \left(\sum_{k=1}^{q} u_k \varphi^k(\xi^e) \right) \\
&\text{so that} \sum_{k=1}^{q} u_k \int_{\mathbb{R}} \varphi^k(\xi) d\xi = 1 \\
&\sum_{k=1}^{q} u_k \varphi^k(\xi) \geq 0, \quad \forall \xi \in \mathbb{R}, \\
&\sum_{k=1}^{q} u_k \varphi^k \in A \quad u_k \in \mathbb{R},\ k = 1, \ldots
\end{aligned}
\tag{6}
$$

where we take the function $\sum_{k=1}^{q} u_k \varphi^k(\xi^e)$ to be the orthonormal basis defined by $\sum_{k=1}^{q} u_k \varphi^k(\xi^e) \equiv u_0 + \sum_{k=1}^{q} (u_k \cos(q\pi V^{\bar{t},e}) + u_{k+q} \sin(q\pi V^{\bar{t},e}))$ and where $V^{\bar{t},e}$ is a function of outcome variables; for example, $V^{\bar{t},e} \equiv TA^{\bar{t},e}$ or the total portfolio wealth at the terminal period $\bar{t}$. This function is explicitly integrated to give the second equation of (6). We use $q = 3$, which was very satisfactory for the applications we did.

Let $a = \min_{e \in X^{\bar{t}}} V^{\bar{t},e}$ and $b = \max_{e \in X^{\bar{t}}} V^{\bar{t},e}$. Then the density function is defined over the interval $[a, b]$ or perhaps a slightly enlarged interval to accommodate tails. The non-negativity

[9] There are several different indicators or measures that may be used and applied in our framework as objectives or as CVaR constraints. Some references where these indicators are discussed include Bank for International Settlements BIS (2002), Asian Development Bank ADB (2001), World Bank and International Monetary Fund (2001a), and International Organization of Securities Commissions IOSCO (2002) for both debt and reserve management.

constraint is handled by dividing the interval into many sub-intervals and constraining the function to be non-negative at each of the interval points. One or two thousand intervals are easily handled within GAMS using the nonlinear solver CONOPT or CPLEX with the quadratic optimizer.

We now discuss the shaping of density functions within the model. Once the model is solved (say under the objective of maximizing the expected return at the end of the horizon), the density functions for all the factors under consideration can be estimated as above. Typically, one will have preferences on the shapes of the density functions. These preferences can be defined formally as objective functions, with or without constraints, as noted above. Another way to state the role of preferences is that they in some sense "shape" the preferred density functions of the outcomes. The advantages of more explicit "shaping" is that, unlike what is typically done in objective functions, one does not need to specify or estimate some parameters, such as utility preferences. Rather, one allows the policy makers to review the actual density functions obtained and then state their preferences as criteria related to concepts such as "a less fat left-tail" or "less probability mass in this or that region."

The actual techniques applied to change the shape of densities to fit the objectives of the policy makers can vary. We may shape the left or the right side of the density and in fact we can specify several points on the densities or distributions that we wish to shape. Furthermore, we may shape several densities or distributions at the same time and several densities or distributions referring to different periods of time.

Mathematically, we can characterize the preferences for the shape of the density as the function given in (7) and (8) for a maximization problem.[10] A slightly different form, discussed later, is given for a minimization problem. The function is like a utility function in that it has a positive first derivative and non-positive second derivative. Therefore in maximizing the function (7), we will be maximizing W^t but the purpose is considerably different. The interval $[p, q]$ is the interval in which we would like to push as much of the probability mass of the density function as possible. The values r_1 and r_2 determine how strongly we wish to push the probability mass into that interval. The important aspect is that the parameters r_1, r_2 and $[p, q]$ are defined explicitly and are not estimated in any way.

$$\text{MAX } E\left[\sum_{t\in T} \delta^t r_1^t (q^t - p^t)\Theta_{\frac{r_1^t}{r_2^t}}\left(\frac{W^t - p^t}{q^t - p^t}\right)\right] \tag{7}$$

$$\Theta_e(\lambda) = \begin{cases} \lambda & \text{for } \lambda \le 0 \\ \lambda - \dfrac{\lambda^2(e-1)}{2e} & \text{for } 0 \le \lambda \le 1 \\ \dfrac{\lambda}{e} + \dfrac{e-1}{2e} & \text{for } \lambda \ge 1 \end{cases} \tag{8}$$

This function is a piece-wise linear-quadratic and can be solved computationally very efficiently.

One example of such an objective for a debt manager is the minimization of a preference function defined over the expected end-of-horizon ratio of total external debt to GDP. The preference function may, for example, be stated so as to push the probability mass of debt/GDP as much as possible below 60% while being rather indifferent to pushing it below 10%. The

[10] The reason for using these preference functions is discussed in Claessens et al. (1998).

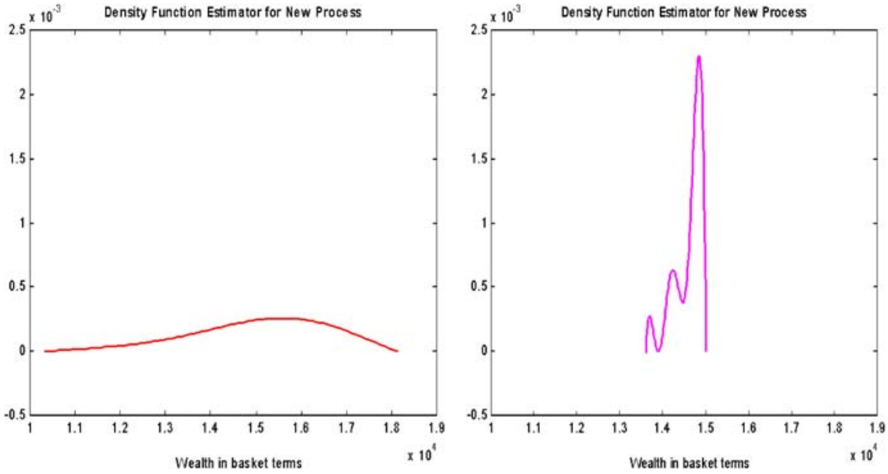

Fig. 2 Density functions before and after shaping

60% value is perhaps determined by policy, and thus relatively hard, while the 10% may be a softer, preference figure. Viewing the outcome of the density function, however, the policy makers may decide to make the 10% target a harder target, as the "costs" of doing so may be relatively low. The point is that the policy makers can observe the outcome and then decide in which way to shape the density function.

Figure 2 is an example of the density functions of the reserves portfolio of a central bank at a point in the future. The density on the left is the future portfolio value, measured in 10's of billions of currency basket units made up primarily of US dollars, Euros, and Yen for an initial set of decisions. It was obtained by maximizing the expected value of the portfolio at the horizon measured in the currency basket. The density on the right hand was obtained after visualizing the density on the left hand and then deciding to squish its shape by applying a preference function to push as much of the probability mass of the portfolio wealth into a range between 14 to 15 billion of basket currency units. To achieve the reshaping, there was a redistribution of the optimal currency and duration of the portfolio. The process becomes thus very intuitive: define an interval where much of the probability mass is preferred to lie, visualize and evaluate the result, and finally obtain the outcome in terms of the new portfolio composition.

Another important measure for shaping distributions or densities is conditional value-at-risk (CVaR). CVaR is equivalent to the expected shortfall of a target at a specified level probability or confidence level $\alpha\%$. Mathematically, it is the concept of the average loss in the worst $\alpha\%$ cases.[11] CVaR is related to value-at-risk (VaR) and for loss distributions, CVaR is always greater than or equal to VaR. Therefore, if we put a constraint on CVaR (an upper bound), the VaR value is also constrained to be below that value. CVaR may be thought of

[11] This is explained very precisely mathematically in Acerbi and Tasche (2002). This paper also discusses the relationship between CVaR, expected shortfall, worse conditional expectation, tail conditional expectation, and VaR. Since we apply CVaR in discrete distributions, a theoretical correction needs to be applied in differentiating between a continuous and a discrete CVaR as we use here. That is discussed in detail in Rockafellar and Uryasev (2002). We think it is not necessary to discuss these details here as it would unnecessarily complicate matters and not affect the computational results. However, we point it out as it affects the precise definition of the terms discussed in Acerbi and Tasche (2002).

as a more robust measure than VaR, since it provides the average loss for those $\alpha\%$ of cases rather than the minimum loss, and therefore takes into account more fully when losses can be extreme.

The CVaR has some useful properties for modeling purposes. Specifically, CVaR constraints can be modeled as linear constraints whereas VaR constraints are non-convex and thus more complex to model and solve. This makes CVaR constraints the method of choice for shaping the distributions or densities for dynamic stochastic optimization models, especially when the models are also very large. Furthermore, since CVaR always bounds the VaR measure, it is possible to get distributions that are VaR constrained by sequentially relaxing the CVaR constraint. A more detailed discussion of CVaR constraints in general can be found in Rockafellar and Uryasev (2002). The equations in our notation for a loss function f are given in (9). Specific examples are given in the next section.

$$\begin{aligned} & z_e \geq f_e(x) - \alpha, z_e \geq 0, \quad \text{and} \\ & \sum_{e \in X^t} \pi^{t,e} z_e \leq \rho(C_\rho - \alpha) \quad \forall e \in X^t, \text{for some } t \end{aligned} \tag{9}$$

The CVaR concept allows us to translate intuitive descriptions of constraints on distributions into mathematical descriptions for use in our modeling framework. One example of a CVaR constraint can be that in the 1% worse cases average losses should not exceed 10% of the portfolio value. Another one could be that in the 5% worse cases the average shortfall in liquidity should not exceed 20%. And, as mentioned, since we can incorporate VaR type constraints in this framework using relaxations of the CVaR constraints, a VaR type constraint could be that with a 95% confidence level the external debt to GDP ratio is less than 60%.

We may apply several of these constraints at one time to one single distribution. We may, for example, constrain a distribution of returns to certain loss levels at the 50% confidence level, the 85% confidence level, and the 99.9% confidence level. We can also constrain several distributions at the same time. In doing so, however, we may impose excessively tight constraints that could mean there is no feasible solution anymore. This is why a two-stage process may be necessary that first attempts to find a feasible solution to all constraints that have been imposed (or at least a solution that is as close as possible given the constraints). If the solution is only close, then the close solution can be accepted or the constraints can be relaxed. Having found a feasible solution, the procedure can then continue to find an optimal solution.

There are other ways of shaping distributions through general and hard constraints. For example, one may wish to have no unrealized loss of principal over a specified period. This constraint can be applied directly and the result will be to cut off the density function of returns at zero over the period considered. Another example is to constrain the expected value of returns and the downside returns to some specific levels. The disadvantage of these general and hard constraints is that they can be quite costly. To assure, for example, no chance of loss whatsoever may imply a very conservative portfolio composition with a very low expected rate of return. In practice, there is likely more flexibility, with management willing to take an acceptable level of risk of a minor loss of say 1%. For this reason, the technology of shaping densities is closer to the problem most managers will face.

Lastly, to shape the density functions, one can expand the set of investments and/or allow for the use of derivatives. One can use a derivative, for example, to flatten the density function, that is, to reduce "risk". However, since derivatives may have impacts on the portfolio as a whole—it may for example reduce the expected return or other expected values—one will need to evaluate their use in the overall context. One can do so easily under this framework by adding the derivative to the model as another instrument or decision variable. Derivatives are especially easy to add since their pricing and volatility are determined by the parameters of the underlying processes, which will already be included in the modeling. In this case it may be desirable to solve the moment matching problem using the volatility values implied by the current derivative prices.

3 Specific reserves management problems

Several of the techniques discussed can be applied simultaneously and one will probably do so in order to obtain a solution that satisfies the various criteria as they are typically defined, explicitly or implicitly, by a particular central bank or government of safety, liquidity, returns, and stability.[12] In this section, some specific examples as applied to commonly stated foreign exchange reserves management problems are discussed. We start with a macro-economic approach that focuses on managing the ratio of reserves to short-term-debt. Next, we look at a coordinated approach between the central bank and the ministry of finance to satisfy sovereign issues. Lastly, we consider improving the rates of return on foreign exchange reserves while satisfying several additional constraints.

3.1 A macro-economic approach

As an example how our framework can be applied in the case of foreign exchange reserves risk management when macro-economic issues are most important, we consider the objective of managing the ratio of debt payments falling due relative to foreign exchange reserves. In this case, we assume it is desirable to manage short-term external debt relative to reserves so that debt payments falling due within the next 12 months do not exceed foreign exchange reserves. We explore two specific possibilities of constraints. In the first, we assume that the central bank manages its reserves level and needs to obtain benchmarks for reserves' currency and asset class composition but has no influence over variables regarding public debt. In this case we assume short-term-debt is exogenous from the point of reserves management, although it may be dependent on other variables such as trade and GDP. Given this problem, we formulate the preference function following the previous convention. The objective to manage the ratio of reserves to short-term-debt can be defined to apply at some point in the future (the horizon) only or as a weighted sum of the ratio at all intermediate points in time. The latter case would allow for some smoothing of the ratio over time, say if there was an unexpected temporary shortfall. This preference function is formulated in Eqs. (14), (15), and (16).

Since short-term-debt is assumed to be exogenous, but still stochastic, maximizing the preference function has the effect of pushing much of the probability mass of the ratio of reserves to short-term-debt into the interval range $[p, q]$, while maximizing the value of reserves. Here we will let $[p, q] = [1, 1.05]$. The function has the effect of trying to manage reserves to best meet the targeted ratio, but it allows for uncertainty. This could result in some volatility in returns and one may want to add some constraints restricting this volatility. This could take the form of a conditional value-at-risk constraint formulated as follows: in 1% of worst cases the average shortfall in returns below 3% should not exceed 2%. This constraint is formulated in Eq. (19).

[12] Stability is the orderly transition over time of specified factors such as the ratio of reserves to short-term-debt, ratio of debt to GDP, portfolio rebalancing, liquidity requirements, etc.

We can add other constraints as well such as maintaining a minimum amount of foreign exchange reserves to meet expected liquidity requirements necessary for intervention. We may want to estimate these liquidity requirements from historical data and generate future stochastic requirements exogenously and/or dependent on factors such as GDP and terms of trade shocks. We would then set aside a portion of our portfolio for liquidity such that, for example, in 2% of the worst cases the average shortfall in the level of the liquidity portfolio should not exceed 10%. The model will then determine the level of this liquidity portfolio, while also meeting the other constraints and objectives. This constraint is formulated in Eqs. (20). These examples of objectives and constraints are summarized in the following Table 1.

The notation for the equations that follow is done in such a way that it can be translated directly into GAMS. The notation $\langle t = 0\rangle$ is used to denote that the expression preceding it only appears for the case when t is zero. The notation $\sum_{(t-1,f)\in AT^{t,e}} A_{i,c}^{\tau,\varepsilon,t-1,f}$ is used to refer to the amount of holdings of asset i in currency c acquired in time τ and event ε that are held in time period $t-1$ and event f. The summation is just used to pull out the correct event f in the previous time. This notation translates directly into that used in GAMS.

The first Eq. (10) defines the cash flows in each time period, for each event, and in each currency. It balances all cash flows by currency.

$$
\begin{aligned}
&\underset{\text{[new assets]}}{\sum_{i\in I} A_{i,c}^{t,e,t,e}} + \underset{\left[\begin{array}{l}\text{transaction costs}\\ \text{on new assets}\end{array}\right]}{\sum_{i\in I} tca_i A_{i,c}^{t,e,t,e}} + \underset{\left[\begin{array}{l}\text{transfer of}\\ \text{currency } c \text{ to } d\end{array}\right]}{V_{c,d}^{t,e}} + \underset{\left[\begin{array}{l}\text{cash out/in}\\ \text{in currency } c\end{array}\right]}{cash_c^{t,e}}\\
&\underset{\text{[liquidity portfolio]}}{+ LQ^{t,e}\langle \text{if } c = \text{“}USD\text{”}\rangle} = \underset{\text{[return on assets]}}{\sum_{i\in I}\sum_{(\tau,\varepsilon)\in AT^{t,e}} \rho_{i,c}^{\tau,\varepsilon} \sum_{(t-1,f)\in AT^{t,e}} A_{i,c}^{\tau,\varepsilon,t-1,f}}\\
&\underset{\text{[return on liquidity portfolio]}}{+ \sum_{(t-1,\varepsilon)\in AT^{t,e}} \left(1+\rho^{t-1,\varepsilon}_{\text{“}short\text{”},\text{“}USD\text{”}}\right) LQ^{t-1,\varepsilon}} + \underset{\text{[initial portfolio]}}{\sum_{i\in I} ina_{i,c}\langle \text{if } t = 0\rangle}\\
&\underset{\text{[initial liquidity portfolio]}}{+ liqb\,\langle \text{if } t=0 \text{ and } c = \$\text{US}\rangle} + \underset{\text{[asset sales]}}{\sum_{i\in I}\sum_{\substack{(\tau,\varepsilon)\in AT^{t,e}\\ \tau\ge t-mat_i}}\sum_{(t-1,f)\in AT^{t-1,f}} (1+tcs_i)\eta_{i,c}^{\tau,t,e}\left(A_{i,c}^{\tau,\varepsilon,t-1,f} - A_{i,c}^{\tau,\varepsilon,t,e}\right)}\\
&\underset{\text{[transfers of currency } d \text{ to } c\text{]}}{+ \sum_{d\in C, d\ne C} \frac{tc_c \gamma_c^{t,e} V_{d,c}^{t,e}}{\gamma_d^{t,e}}} + \underset{\text{[matured assets]}}{\sum_{i\in I}\sum_{(\tau,\varepsilon)\in AT^{t,e}}\sum_{(t-1,f)\in AT^{t,e}} A_{i,c}^{\tau,\varepsilon,t-1,f}}\\
&\qquad\qquad\qquad\qquad\qquad \forall c\in C,\ t\in T, e\in X^t
\end{aligned}
\tag{10}
$$

The next equation is necessary to ensure that the definition of the sales of assets is properly defined in the model to be monotonic.

$$
\sum_{(t-1,f)\in AT^{t,e}} \left(A_{i,c}^{\tau,\varepsilon,t-1,f} - A_{i,c}^{\tau,\varepsilon,t,e}\right) \ge 0 \tag{11}
$$

[monotonicity constraint so that sales are reflected in changes in holdings]

$$\forall\, i, c, t, e, \tau, \ni(\tau,\varepsilon)\in AT^{t,e} \text{ and } (t-\tau) < mat_i$$

Table 1 Summary of main model equations for macro-economic approach

Macro issue: Managing reserves relative to short-term debt.			
Form	Type	Criteria	Description
Objective maximize (16)	Preference on $\frac{reserves}{short\text{-}term\text{-}debt}$	Stability of ratio & returns	Shape density of outcomes so that ratio most likely falls into the range [1,1.05] while maximizing reserves.
Risk constraint (18)	Bound on returns	Safety of principal	Limit the unrealized losses so the portfolio return is always ≥ 0 as measured in the basket currency.
Risk constraint (19)	CVaR constraint on returns	Safety on extreme losses	In 1% of worst cases the average shortfall in returns below 3% should not exceed 2%.
Risk constraint (20)	CVaR constraint on liquidity	Liquidity	In 2% of worse cases the average shortfall in the liquidity portfolio should not exceed 10%.

The next equation constrains the minimum activity in any currency. This may be a desirable policy constraint for various currencies.

$$\begin{aligned} &\underset{\text{[minimum activity as \% of total]}}{ma_c^t TA^{t,e}\gamma_c^{t,e}} \quad \leq \quad \underset{\text{[liquidity portfolio]}}{LQ^{t,e}} \langle \text{if } c = \text{“USD”}\rangle \\ &+ \underset{\text{[new assets]}}{\sum_{i\in I} A_{i,c}^{t,e,t,e}} + \underset{\text{[assets marked to market]}}{\sum_{i\in I}\sum_{(\tau,\varepsilon)\in AT^{t,e}} \eta_{i,c}^{\tau,t,e} A_{i,c}^{\tau,\varepsilon,t,e}} \\ &\qquad \forall c, t, e \ni e \in X^t \end{aligned} \tag{12}$$

The next constraint defines the total assets valued in the numeraire of the portfolio.

$$\begin{aligned} \underset{\text{[total assets]}}{TA^{t,e}} = \underset{\text{[new assets]}}{\sum_{i\in I, c\in C} \frac{A_{i,c}^{t,e,t,e}}{\gamma_c^{t,e}}} + \underset{\text{[assets marked to market]}}{\sum_{i\in I, c\in C}\sum_{(\tau,\varepsilon)\in AT^{t,e}} \frac{\eta_{i,c}^{\tau,t,e} A_{i,c}^{\tau,\varepsilon,t,e}}{\gamma_c^{t,e}}} + \underset{\text{[liquidity portfolio]}}{LQ^{t,e}} \\ \forall t, e \ni e \in X^t \end{aligned} \tag{13}$$

The next constraint begins the definition of the preference function. It allocates value to the three segmentations of the preference function according to (15) so that they are assured their proper values due to the concavity of the function under maximization. In this particular case we only define it at the horizon $t = \bar{t}$.

$$\frac{TA^{t,e}}{std^{t,e}} - p^t = (q^t - p^t)\left(SG_1^{t,e} + SG_2^{t,e} + SG_3^{t,e}\right) \quad \forall t = \bar{t}, e \in X^{\bar{t}}. \tag{14}$$

The next set of constraints ensures that the segmentations take their proper values and define the functions given in (7) and (8).

$$SG_1^{t,e} \leq 0, \quad 0 \leq SG_2^{t,e} \leq 1, \quad 0 \leq SG_3^{t,e} \quad \forall t = \bar{t}, e \in X^{\bar{t}} \tag{15}$$

The next constraint is the definition of the value of the preference function that is built up from the three segmentations. We only define it for $t = \bar{t}$ and therefore set $\delta^{\bar{t}} = 1$.

$$PRF = \sum_{t=\bar{t}, e \in X^{\bar{t}}} \pi^{t,e} \delta^t r_1^t (q^t - p^t) \left(SG_1^{t,e} + SG_2^{t,e} - \left(SG_2^{t,e}\right)^2 \left(\frac{r_1^t}{r_2^t} - 1 \right) \frac{r_2^t}{2r_1^t} + SG_3^{t,e} \frac{r_2^t}{r_1^t} \right) \tag{16}$$

The following defines the total expected value of the portfolio as measured in the basket currency.

$$TWB^t = \sum_{e \in X^t} \pi^{t,e} \frac{TA^{t,e}}{bas^{t,e}} \quad \forall t \in T \tag{17}$$

The following constrains the return on the portfolio as measured in the basket currency to have a lower bound. We set $nret^t = 0$ in our example.

$$nret^t \sum_{(t-1,\varepsilon) \in AT^{t,e}} \frac{TA^{t-1,\varepsilon}}{bas^{t-1,\varepsilon}} \le \frac{TA^{t,e}}{bas^{t,e}} - \sum_{(t-1,\varepsilon) \in AT^{t,e}} \frac{TA^{t-1,\varepsilon}}{bas^{t-1,\varepsilon}} \quad \forall t, e \ni e \in X^t. \tag{18}$$

The following defines the CVaR constraint of the safety on returns for extreme losses so that in 1% of the worst cases the average shortfall in returns below 3% should not exceed 2% with respect to the numeraire currency.

$$\begin{aligned} ZRET^{t,e} &\ge \frac{cret_3^t - TA^{t,e}}{cret_3^t} - \mathrm{A}_{ret}^t, \quad ZRET^{t,e} \ge 0 \\ \sum_{e \in X^t} \pi^{t,e} ZRET^{t,e} &\le cz \left(lz - \mathrm{A}_{ret}^t \right) \quad \text{For } t = \bar{t}, e \in X^t \end{aligned} \tag{19}$$

where we set $cz = .01$ and $lz = .01$ (3%–2%).

The following defines the CVaR constraint on the liquidity portfolio so that in 2% of the worst cases the average shortfall in the liquidity portfolio should not exceed 10%.

$$\begin{aligned} ZLIQ^{t,e} &\ge \frac{cliq^{t,e} - LQ^{t,e}}{cliq^{t,e}} - \mathrm{A}_{liq}^t, \quad ZLIQ^{t,e} \ge 0 \\ \sum_{e \in X^t} \pi^{t,e} ZLIQ^{t,e} &\le cz \left(lz - \mathrm{A}_{liq}^t \right) \quad \text{For } t = \bar{t}, e \in X^t \end{aligned} \tag{20}$$

where we set $cz = .02$ and $lz = .1$ (10%).

The problem solved then is to maximize PRF subject to (10) through (20). The constraint (18) will not cause an infeasible solution as long as there are government bonds available with maturities less than the time between periods. A bond in the basket currency can then be structured from the bonds in each currency. The constraint (19) will not cause an infeasible solution as long as there are bonds of sufficient return. If not, then the CVaR values should be adjusted. Constraint (20) is also unlikely to cause infeasibility. Infeasibilities are not impossible but they are rare in this model.

It may seem as though the model will be very large. This need not be the case because of the way that the sparse tree is generated as is discussed in Kreuser (2002).

3.2 A macro-economic approach with coordination of CB and MoF

In the second case, we introduce decision variables for both debt and reserves management and assume that the central bank (CB) and the ministry of finance (MoF) coordinate their actions to some degree. We now disaggregate both reserves and debt into appropriate classes and currencies and allow both sets of variables as decision variables. We cannot apply the previous preference function in this case as it would lead the model to set short-term-debt to zero as debt is now also a decision variable. Instead, we apply an alternative minimizing preference function of $\frac{TB-TA}{Y}$, where *TB* is the total debt, *TA* is the total amount of foreign exchange reserves, and *Y* is a scaling variable, in this case assumed to be exports (as the external debt to exports ratio is often used as an indicator of the repayment capacity of a country). This function is defined in (21) and (22). The interval $[p,q]$ is defined appropriate to the individual country. Debt accumulation and reserves levels will be related to each other through the budget constraints of the government and the central bank. Net new debt accumulation will now be determined by the model in such as way as to manage the country's liquidity needs as well as manage the debt to export ratio. The model will also determine the currency and maturity structure of debt and the structure and currency of foreign exchange reserves.

$$\text{MIN}\, E\left\{ r^T(q^T-p^T)\Theta\left[\frac{\frac{TB^T-TA^T}{Y^T}-p^T}{q^T-p^T}\right]\right\} \tag{21}$$

$$\Theta(\lambda)=\begin{cases}0 & \text{if } \lambda\le 0\\ \dfrac{\lambda^2}{2} & \text{if } 0\le\lambda\le 1\\ \lambda-\dfrac{1}{2} & \text{if } \lambda\ge 1\end{cases} \tag{22}$$

In this case, minimizing the preference function has the effect of seeking low borrowing costs and high return on reserves. However, we did not introduce the volatility of returns or costs directly into the preference function. We can do so easily by introducing conditional value-at-risk constraints for both returns and costs, that limit the downside risks for returns and the upside risks for costs. Or, instead of introducing conditional value-at-risk constraints, we can specify explicit constraints on costs and returns. We can also introduce an explicit constraint on short-term-debt, for example, such that short-term-debt is always less than foreign exchange reserves. Alternatively, this can be formulated as a conditional value-at-risk constraint, such as: in the worst 5% of cases the average shortfall in reserves below short-term-debt should not exceed 300 million. The advantage of the latter constraint is that in extreme circumstances reserves may be allowed to fall below short-term debt. These constraints are summarized in Table 2.

Additional constraints to satisfy policy objectives or limits can also be introduced. We can, for example, introduce constraints on the volatility of transaction costs or on the percentage of the portfolio rolled over, either/both on reserves and/or on debt. Legal constraints on the type of investments may also be introduced. The results will be reserves and debt allocations satisfying the required constraints, e.g., on safety, liquidity, returns, and stability. In all cases, the reserves and debt variables will again be disaggregated into maturity and currency classes, thus allowing investment benchmarks to be determined as outcomes of the optimization procedure.

Table 2 Summary of main model equations for macro-economic approach II

Macro issue: Managing debt and foreign exchange reserves			
Form	Type	Criteria	Description
Objective minimize (33)	Preference on $\frac{TB-TA}{Y}$	Stability of ratio and costs & returns	Shape density of outcomes so that ratio most likely falls into the range $[\alpha, \beta]$ while reducing costs and increasing returns.
Risk constraint (34)	CVaR constraint on liquidity	Liquidity	In 2% of worse cases the average shortfall in our liquidity portfolio should not exceed 10%.
Risk constraint (35)	CVaR constraint on short-term-debt	Stability of short-term-debt	In 5% of worst cases the average shortfall in reserves below short-term-debt should not exceed 300 million.
Risk constraint (36)	CVaR constraint on returns	Returns	In 5% of worst cases the average shortfall in returns below 5% should not exceed 2.5%.

The specific preference function chosen depends on each particular situation and the relevant indicator. Alternative indicators appropriate to specific situations, as discussed in IMF (2000), include the ratios of: reserves to short-term external debt; reserves to imports; reserves to broad money; external debt to exports; external debt to GDP; and measures such as the average interest rate on external debt; the average maturity of debt or reserves; and the share of foreign currency external debt in total debt. We do not advocate one particular indicator over another, and many may be used at the same time, but rather emphasize that this framework can be used to incorporate many or even all of these indicators.

The following equations describe the debt model. The reserves and debt models are linked through the objective function and by the variable $CB_c^{t,e}$, the reserves profit and loss variable. The use of the notation $\langle t \neq 0 \rangle$ in the equations means that the expression preceding it only applies when t is not equal to zero.

$$
\begin{aligned}
&\underbrace{\sum_{j\in J} B_{j,c}^{t,t,e}}_{\text{[new borrowings]}} + \underbrace{\sum_{d\in C, d\neq C} \frac{tc_c \gamma_c^{t,e} V_{d,c}^{t,e}}{\gamma_d^{t,e}}}_{\text{[transfers of currency } d \text{ to } c]} \\
&+ \underbrace{CB_c^{t,e}}_{\text{[central bank profit/loss]}} = \underbrace{\sum_{j\in J} tcl_j B_{j,c}^{t,t,e}}_{\text{[new borrowings transaction costs]}} \\
&+ \underbrace{\sum_{j\in J} \sum_{(\tau,\varepsilon)\in AT^{t,e}} \sum_{(t-1,f)\in AT^{t,e}} \left(\alpha_{j,c}^{\tau,\varepsilon} + \beta_j^{t-\tau}\right) B_{j,c}^{\tau,t-1,f} \langle t \neq 0 \rangle}_{\text{[interest and principal on borrowing]}} \\
&+ \underbrace{V_{c,d}^{t,e}}_{\text{[transfer of currency } c \text{ to } d]} + \underbrace{\phi^{t,e} \langle \text{if } c = \text{“local currency”} \rangle}_{\text{[fiscal deficit]}} \\
&+ \underbrace{\sum_{j\in J} \sum_{(t-1,\varepsilon)\in AT^{t,e}} \left(\alpha 0_{j,c} + \beta 0_{j,c}^{t-1}\right) BL_{j,c}^{t-1,\varepsilon} \langle t \neq 0 \rangle}_{\text{[interest and principal on original borrowing]}}
\end{aligned}
\tag{23}
$$

$$+ \ (1+pnb)\sum_{j\in J}\sum_{(\tau,\varepsilon)\in AT^{t,e}}\sum_{(t-1,f)\in AT^{t,e}}\left(1-\beta_j^{t-\tau}\right)\left(B_{j,c}^{\tau,t-1,f}-B_{j,c}^{\tau,t,e}\right)\langle t\neq 0\rangle$$

[prepayments on borrowing]

$$+(1+pnb)\sum_{j\in J}\left[\left(inl_{j,c}-BL_{j,c}^{t,e}\right)\langle t=0\rangle+\sum_{(t-1,\varepsilon)\in AT^{t,e}}\left(1-\beta 0_j^{t-1}\right)\left(BL_{j,c}^{t-1,\varepsilon}-BL_{j,c}^{t,e}\right)\right]$$

[prepayments on old borrowing]

$$\forall c\in C, t\in T, e\in X^t$$

The variable $CB_c^{t,e}$ is also added to the left-hand side of Eq. (10).

The following equation is used to enforce the monotonicity of the borrowing variable $B_{j,c}^{\tau,t,e}$ allowing any part of the borrowing to be prepaid. These equations are defined for new borrowings and for old borrowings or borrowings in place at the beginning of the time period.

$$\sum_{(t-1,\varepsilon)\in AT^{t,e}}\left(\left(1-\beta_j^{t-\tau}\right)B_{j,c}^{\tau,t-1,\varepsilon}-B_{j,c}^{\tau,t,e}\right)\geq 0 \quad \forall j,c,\tau,t,e\ni(\tau<t)\,\text{and}\,(t-\tau)<mat_j$$

[monotonicity constraint to reflect prepayments and principal]

$$\left(inl_{j,c}-BL_{j,c}^{t,e}\right)\langle t=0\rangle+\sum_{(t-1,\varepsilon)\in AT^{t,e}}\left(\left(1-\beta 0_{j,c}^{t-1}\right)BL_{j,c}^{t-1,\varepsilon}-BL_{j,c}^{t,e}\right)\geq 0 \tag{24}$$

[monotonicity constraint to reflect old prepayments and principal]

$$\forall j,c,\tau,\bar{t},e\ni(1<t)\,\text{and}\,(t-1)<mat0_{j,c}$$

$$BL_{j,c}^{0,0}\leq inl_{j,c} \qquad \text{[initialization of borrowings]}$$

The next equation is used to describe a rule to distribute a percentage of the excess profits or losses of the reserves portfolio to the variable $CB_c^{t,e}$.

$$\sum_{c\in C}\frac{CB_c^{t,e}}{\gamma_c^{t,e}}=lcb\left(TA^{t,e}-rcb^t\sum_{(t-1,\varepsilon)\in AT^{t,e}}TA^{t-1,\varepsilon}\right) \tag{25}$$

[definition of central bank profit/loss]

The next equation is used to impose a minimum market activity in each currency. The value of mb_c is a policy variable set by each central bank.

$$mb_c \leq$$

[minimum market activity in currency c]

$$\frac{1}{\gamma_c^{t,e}}\sum_{j\in J}\left(B_{j,c}^{t,t,e}+\sum_{\tau|t-\tau<mat_j}B_{j,c}^{\tau,t,e}+BL_{j,c}^{t,e}\right)\quad\forall c,t,e\ni e\in X^t \tag{26}$$

[outstanding borrowings]

The next equation imposes an upper limit on the total outstanding debt in the local currency. This is assumed to be a function of the market capacity.

$$\begin{array}{ccc} mx^t & \geq & \sum_{j\in J}\left(B^{t,t,e}_{j,\tilde{c}} + \sum_{\tau|t-\tau<mat_j} B^{\tau,t,e}_{j,\tilde{c}} + BL^{t,e}_{j,\tilde{c}}\right) \\ \left[\begin{array}{l}\text{maximum market activity}\\ \text{in local currency } \tilde{c}\end{array}\right] & & \left[\begin{array}{l}\text{outstanding borrowing in}\\ \text{local currency } \tilde{c}\end{array}\right] \\ & & \forall\, t, e \ni e \in X^t \end{array} \tag{27}$$

The next equation imposes an upper limit on new external debt. The value of the variable $exmx^{t,e}$ can be set based upon other variables determined by the tree. It can therefore be used to simulate situations when new external debt becomes restricted in times of crises.

$$\begin{array}{ccc} exmx^{t,e} & \geq & \sum_{j\in J}\sum_{c,c\neq\tilde{c}} \frac{1}{\gamma^{t,e}_c}\left(B^{t,t,e}_{j,c}\right) \\ \left[\begin{array}{l}\text{maximum market activity}\\ \text{for external debt}\end{array}\right] & & [\text{new external debt}] \\ & & \forall\, t, e \ni e \in X^t \end{array} \tag{28}$$

The following equation imposes an upper limit on new local debt. The variable $ldmx^{t,e}$ can be determined in a similar way as $exmx^{t,e}$.

$$\begin{array}{ccc} ldmx^{t,e} & \geq & \sum_{j\in J} \frac{1}{\gamma^{t,e}_{\tilde{c}}} B^{t,t,e}_{j,\tilde{c}} \quad \forall\, t, e \ni e \in X^t \\ \left[\begin{array}{l}\text{maximum market}\\ \text{activity for local debt}\end{array}\right] & & [\text{new local debt}] \end{array} \tag{29}$$

The next equation defines the total amount of external debt in the numeraire currency.

$$\begin{array}{cc} TB^{t,e} & = \sum_{j\in J}\sum_{c,c\neq\tilde{c}} \frac{1}{\gamma^{t,e}_c}\left(B^{t,t,e}_{j,c} + \sum_{\tau|t-\tau<mat_j} B^{\tau,t,e}_{j,c} + BL^{t,e}_{j,c}\right) \quad \forall t, e \epsilon\, e \in X^t \\ \left[\begin{array}{l}\text{total external}\\ \text{debt}\end{array}\right] & [\text{outstanding external debt}] \end{array} \tag{30}$$

The next constraint begins the definition of the preference function. It allocates values to the three segments as in (32), such that their proper values are achieved, in this case, by the convexity of the function under minimization.

$$\frac{TB^{t,e} - TA^{t,e}}{exports^{t,e}} - p^t = (q^t - p^t)\left(SG^{t,e}_1 + SG^{t,e}_2 + SG^{t,e}_3\right)$$
$$\text{and } [p^{\bar{t}}, q^{\bar{t}}] = [\alpha, \beta] \quad \forall t = \bar{t}, e \in X^{\bar{t}} \tag{31}$$

The next set of constraints ensures that the segments SG that define the objective in (31) take their proper values.

$$SG^{t,e}_1 \leq 0, \quad 0 \leq SG^{t,e}_2 \leq 1, \quad 0 \leq SG^{t,e}_3 \quad \forall t = \bar{t}, e \in X^{\bar{t}} \tag{32}$$

The next constraint is the definition of the value of the preference function that is built up from the three segments. We note that in this particular case we use only the last period $t = \bar{t}$ and therefore set $\delta^{\bar{t}} = 1$ since we do no discounting.

$$PRF = \sum_{t=\bar{t}, e \in X^{\bar{t}}} \pi^{t,e} \delta^t r^t (q^t - p^t) \left(\frac{\left(SG_2^{t,e}\right)^2}{2} + SG_3^{t,e} \right) \tag{33}$$

The following defines the CVaR constraint on the liquidity portfolio so that in 2% of the worst cases the average shortfall in the liquidity portfolio should not exceed 10%.

$$\begin{aligned} & ZLIQ^{t,e} \geq \frac{cliq^t - LQ^{t,e}}{cliq^t} - ALPHA^t_{liq}, \quad ZLIQ^{t,e} \geq 0 \\ & \text{and} \sum_{e \in X^t} \pi^{t,e} ZLIQ^{t,e} \leq cz \left(lz - ALPHA^t_{liq} \right) \quad \text{For } t = \bar{t}, e \in X^t \end{aligned} \tag{34}$$

where we set $cz = .02$ and $lz = .1$ (10%).

The following defines the CVaR constraint for the stability of the short-term-debt such that in 5% of worst cases the average shortfall in reserves below short-term-debt should not exceed 300 million.

$$\begin{aligned} & ZRES^{t,e} \geq \sum_{j \in \hat{J}} \sum_{c, c \neq \tilde{c}} \frac{1}{\gamma_c^{t,e}} \left(B^{t,t,e}_{j,c} + \sum_{\tau | t - \tau < mat_j} B^{\tau,t,e}_{j,c} + BL^{t,e}_{j,c} \right) - TA^{t,e} - ALPHA^t_{res}, \\ & ZRES^{t,e} \geq 0, \quad \sum_{e \in X^t} \pi^{t,e} ZRES^{t,e} \leq cz \left(lz - ALPHA^t_{ret} \right) \\ & \qquad \text{For } \hat{J} \equiv \{ j \in J \mid j \text{ is short-term-debt} \}, \text{and } t = \bar{t}, e \in X^t \end{aligned} \tag{35}$$

where we set $cz = .05$ and $lz = 300$.

The following defines the CVaR constraint of the safety on returns for extreme losses so that in 5% of the worst cases the average shortfall in returns below 5% should not exceed 2.5% with respect to the numeraire currency.

$$\begin{aligned} & ZRET^{t,e} \geq \frac{cret_3^t - TA^{t,e}}{cret_3^t} - ALPHA^t_{ret}, \quad ZRET^{t,e} \geq 0, \quad \text{and} \\ & \sum_{e \in X^t} \pi^{t,e} ZRET^{t,e} \leq cz \left(lz - ALPHA^t_{ret} \right) \quad \text{For } t = \bar{t}, e \in X^t \end{aligned} \tag{36}$$

where we set $cz = .05$ and $lz = .025$ (5%–2.5%).

Then the problem that is solved is to minimize *PRF* defined as in (33) subject to the constraints (10) to (12) and (23) to (36). Equation (35) could be seen as a concession to the ministry of finance and Eq. (36) as a concession to the central bank. Various other constraints of a similar nature could be imposed to satisfy each institution separately, while optimizing a sovereign objective.

Because of several constraints that have limits on them, especially the market constraints (28) and (29) in times of crises, it is possible to introduce infeasibilities. In most if not all cases the infeasibilities can be mitigated by the introduction for each time and event of a "cash" variable into the debt balance sheet Eq. (23). This variable need only be denominated in one currency. This variable can be considered to be contingent external support in the form of an IMF loan, a sale of gold reserves, debt write-off, or debt default. Then the discounted sum of these variables is minimized. Non-zero variables indicate those time periods and events when problems may occur. These variables are then fixed at their minimum level and the resulting model is then optimized in the usual way by optimizing the preference function. We can then measure the likelihood and level of such difficulties.

Since the decision variables include the level, currency, and maturity of debt we can introduce constraints on several kinds of debt measures, including the widely used cost-at-risk and debt-service-at-risk.[13]

3.3 Improving returns on foreign exchange reserves

In this section we assume that the central bank has substantial foreign exchange reserves such that its macro-economic objectives are met. Its main concern rather is assumed to be to improve returns through active management and using various (new) financial instruments, including derivatives. It might choose as its objective function the maximization of the expected value of the terminal level of reserves, where the terminal period may be any period, but typically will be after one or two years. One of the main policy issues is likely what currency the value of foreign exchange reserves and the realized rates of return should be measured in. An approach that is often used is a currency basket with weights equal to the proportion of each currency in the country's imports. When the weights are constant, a basket or reference currency results. In turn the basket is then assumed to represent a risk neutral position vis-à-vis currency risk (i.e., investing in currencies in the same proportion as in the basket represent a risk-neutral investment position). The returns in this model could be volatile, however, so an explicit constraint on the downside returns might be introduced, with the same reference basket currency used to measure the constraint. A limit on transaction costs may also be imposed so that trading does not become excessive. This may take the form of a constraint that in the worst 10% of cases the average total transaction costs in any period should not exceed 10 million. The constraints and objective are described in Table 3 with Eq. (39) representing the constraint on downside returns. The objective is described as a preference function, with two parameters, α and β. With α large, the preference function becomes equivalent to maximizing returns. And changing the size of the interval $[\alpha, \beta]$ can be used to shape the density function.

The model consists of all the Eqs. (10) through (20) plus the equations discussed in the following.

The first constraint begins the definition of the preference function. It allocates value to the three segments of the preference function according to (15) to ensure their proper values due to the concavity of the function under maximization. The total assets are now measured in the basket currency.

$$\frac{TA^{t,e}}{bas^{t,e}} - p^t = (q^t - p^t)\left(SG_1^{t,e} + SG_2^{t,e} + SG_3^{t,e}\right) \quad \forall t = \bar{t}, e \in X^{\bar{t}} \tag{37}$$

[13] Cost-at-risk is discussed in detail in publications from Danmarks NationalBank, from the Swedish National Debt Office, and from the UK Debt Management Office. It is a popular measure also used by other debt offices. Here we not only measure it but also control it by putting constraints on it like CVaR constraints and we do so in the context of optimizing other measures at the same time. There are several other "at-risk" measures that can also be constrained or have their density shaped such as Liquidity-at-risk, Funding-at-risk, Rebalancing-at-risk, Transaction costs-at-risk, etc. These can be shaped at one period or at several periods in time.

Table 3 Equations for improving returns

Micro issue: Maximize returns with liquidity and safety			
Form	Type	Criteria	Description
Objective maximize (37)	Preference on returns	Returns	Shape density of returns measured in the currency basket so that they most likely fall into the range $[\alpha, \beta]$.
Risk constraint (38)	CVaR constraint on liquidity	Liquidity	In 2% of worse cases the average shortfall in our liquidity portfolio should not exceed 10%.
Risk constraint (39)	CVaR constraint on returns	Safety	In 5% of worst cases the average shortfall in returns below 6% should not exceed 2.5%.
Risk constraint (41)	CVaR constraint on transactions	Stability	In 10% of worst cases the average transaction costs beyond 20 million should not exceed 30 million.

The preference function, *PRF,* is then the same as the function (16) and the limits on the segmentation variables, *SG*, are the same as (15).

The following defines the CVaR constraint on the liquidity portfolio so that in 2% of the worst cases the average shortfall in the liquidity portfolio should not exceed 10%.

$$
\begin{aligned}
&ZLIQ^{t,e} \geq \frac{cliq^t - LQ^{t,e}}{cliq^t} - ALPHA^t_{liq}, \quad ZLIQ^{t,e} \geq 0, \quad \text{and} \\
&\sum_{e \in X^t} \pi^{t,e} ZLIQ^{t,e} \leq cz\left(lz - ALPHA^t_{liq}\right) \quad \text{For } t = \bar{t}, e \in X^t
\end{aligned} \tag{38}
$$

where we set $cz = .02$ and $lz = .1$ (10%).

The following defines the CVaR constraint of the safety on returns for extreme losses so that in 5% of the worst cases the average shortfall in returns below 6% should not exceed 2.5%.

$$
\begin{aligned}
&ZRET^{t,e} \geq \frac{cret^t_3 - TA^{t,e}}{cret^t_3} - ALPHA^t_{ret}, \quad ZRET^{t,e} \geq 0 \quad \text{and} \\
&\sum_{e \in X^t} \pi^{t,e} ZRET^{t,e} \leq cz\left(lz - ALPHA^t_{ret}\right) \quad \text{For } t = \bar{t}, e \in X^t
\end{aligned} \tag{39}
$$

where we set $cz = .06$ and $lz = .035$ (6%–2.5%).

The following equation defines the total value of the transaction costs in the numeraire currency.

$$
\begin{aligned}
\underset{\text{[total transaction costs]}}{TRS^{t,e}} = &\sum_{c \in C, c \neq \tilde{c}} \frac{1}{\gamma^{t,e}_c} \sum_{i \in I} \Bigg(tca_i A^{t,e,t,e}_{i,c} + \sum_{d \in C, d \neq c} \frac{tc_c \gamma^{t,e}_c V^{t,e}_{d,c}}{2\gamma^{t,e}_d} \\
&+ \sum_{(\tau,\varepsilon) \in AT^{t,e}} \sum_{(t-1,f) \in AT^{t-1,f}} tcs_i \eta^{\tau,t,e}_{i,c} \left(A^{\tau,\varepsilon,t-1,f}_{i,c} - A^{\tau,\varepsilon,t,e}_{i,c}\right) \Bigg) \\
&\forall t \in T, e \in X^t
\end{aligned} \tag{40}
$$

The following defines the CVaR constraint for the transaction costs such that in 10% of worst cases the average transaction costs beyond 20 million should not exceed 30 million.

$$ZTRS^{t,e} \geq TRS^{t,e} - 20 - Alpha^{t}_{trs}, \quad ZTRS^{t,e} \geq 0, \quad \text{and}$$
$$\sum_{e \in X^t} \pi^{t,e} ZTRS^{t,e} \leq cz\left(lz - ALPHA^{t}_{trs}\right) \quad \text{For } t \in T, e \in X^t \tag{41}$$

where we set $cz = .1$ and $lz = 10$.

The problem solved is then to maximize *PRF* as defined by (16) with (37) replacing (14) and subject to (10) through (12), (15), (16), and (38) through (41). The comments regarding infeasibilities are essentially the same as in Section 3.1.

With more active reserve management, the universe of possible asset variables is typically expanded, including assets of longer duration and lower credit quality. Not only price fluctuation, but also credit risk considerations will then arise. The latter can be handled in the model by simply including asset limits on those assets of lower credit rating, which is what is usually done by central banks. They can also be handled by estimating the stochastic processes driving the actual default risks and incorporating the default risks explicitly in the model. Since the framework is flexible, the difficulty in implementing credit risk arises mainly from the difficulty in determining the stochastic processes for default risks, and not from the technology of incorporating them in the model.

Derivatives may similarly be introduced. Derivatives are essentially combinations of assets and liabilities and therefore in some sense redundant financial instruments. Derivatives can be useful through as they provide for a quick and efficient restructuring of the risk parameters of the portfolio. Derivatives are indexed directly on the risk parameter and require only small investments. An interest futures, for example, provides a payoff directly linked to the underlying interest rate with a very small investment, that is, only an initial margin needs to be put up. Derivatives can also help satisfy easier specific objectives or constraints, such as managing the risk of downside returns, as when options are used with relatively low (or high) exercise prices. Or a commodity priced indexed derivative used to hedge the risk of terms of trade shocks affecting the country's fiscal accounts or banking system stability may be more efficient than the central bank undertaking a direct position in a commodity-price linked investment or liability. More generally, derivatives can allow for alternative types of payoff profiles, importantly including non-linear payoffs, as in the case of options. Derivatives can be used, for example, for meeting risk constraints that require flattening of the density function in relation to returns. And for strategic exchange rate intervention purposes, derivatives can be useful, especially if liquidity constraints are a binding factor (as discussed in Blejer and Schumacher (2000) or if there is a desire to signal a certain policy stance. Derivatives can sometimes also be a cheaper way to manage interest rate, currency or other risks than trading the underlying assets as discussed in Rigaudy (2000). Reflecting these advantages, the use of derivatives by central banks is increasing with the most commonly used being interest and currency forwards and swaps. Although they can be very effective if altering risk profiles in specific ways, derivatives are not meant, or at least should not be used, for leveraging the overall risk of the portfolio.

From a modeling point of view, there is not much difference between how derivatives and other assets and liabilities are treated. The model will just require specifications of the respective payoff functions as they relate to underlying factors. It will consequently choose a derivative over the underlying assets only if they are "cheaper" than the underlying assets themselves and vice-versa. Similarly, through the model the trade-offs between risk and return will be identified and either the derivative or a combination of assets and liabilities most effective in reducing specific risk constraints or increasing return will be chosen.

3.4 Partitioning into active and passive portfolios

A central bank may decide to partition its portfolio into active and passive parts and manage each part separately. We use the words active and passive to mean that the active portfolio is managed more aggressively than the passive part. The two parts may have different sets of rules and instruments and the number of transactions may be higher on the active portfolio than on the passive part. In this case the active portfolio may be managed as discussed in 3.3 and the passive part as in 3.1. Instead of a fixed allocation between the two portfolios, another approach could be to develop a model that determines the optimal proportion of foreign exchange reserves to be allocated to the active portfolio. In order to analyze this situation, we assume that the central bank takes a macro-economic approach as in 3.1. The model would be structured as in 3.1 with the addition of a constraint limiting the volume of asset sales to a level that would be reasonable for the institution to handle. The limit to the level of trades would be chosen to be consistent with the central bank's objectives and ability to manage the institutional environment demands (such as people, back-office and reporting and control functions). A second constraint limiting total transaction costs as in the previous section could also be added. The solution to the model would then give us the current investment choices and the distributions of all future outcomes as well as the decision variables that are contingent on those choices.

With the solution to this model, we will also obtain the density function of the level of trading at any time in the future. Instead of predefining the active and passive portfolios, we can define our active portfolio as consisting of those assets in the overall portfolio that are actively traded or sold and purchased. This then gives us a natural split between an active and a passive portfolio, with the size of the active portfolio and choice of assets that make up the active portfolio determined so as to satisfy the overall objectives and other constraints. The result is an optimal portfolio whereby the active part has been determined consistent with the central bank's objectives and ability to manage trading within an uncertain future environment and consistent in terms of constraints and objectives across both active and passive parts of the portfolio.

4 Application of the framework to foreign exchange reserves management

In this section we discuss creating the appropriate analytical framework, making decisions based upon multiple density functions, and some examples for specific central banks. The procedures for developing the baseline case, presenting results and creating benchmarks, translating into operations and control, stress tests, policy comparisons and analysis, and institutional considerations are discussed in more detail in Claessens and Kreuser (2004).

4.1 Creating an appropriate analytical framework

The first step is a thorough review of the central bank's current operations, investment classes, policies, legal and operational constraints, and requirements for foreign exchange reserves management in terms of safety, liquidity, returns, and stability. This can be followed by a review of how the central bank currently handles risks with a subsequent discussion on how it might handle risks based upon a more formal risk control framework. For example, it may be determined that the important objectives include: getting a reasonable level of returns, reducing the probability of negative unrealized losses, maintaining a satisfactory level of liquidity, reducing the necessity of active portfolio turnover, and seeking a reserves level that keeps as best as possible the ratio of reserves to short-term external debt above a value of one. Such a review and discussion should take into account the absorptive capacity of the central bank staff and resources available in terms of analytical modeling. This process itself is very valuable as it forces the review of what may be many implicit policies and constraints. The review of objectives and requirements can then be translated into objective functions and constraints and on that basis a prototype model along the lines discussed above can be build.

4.2 Example of multiple density shaping

In order to understand how the density shaping process might proceed, it is instructive to look at an actual example. Figure 3 illustrates the shaping of four density functions for reserves simultaneously. It replicates a panel used by the software system **RisKontroller**.[14] The four variables analyzed are wealth measured in dollars, wealth measured in the currency basket, the portfolio average maturity, and the dollar rate of return on assets. This is a simplified example from an actual central bank study with many assumptions similar to those outlined above and using actual simulations, although done a few years ago. The first column depicts the four density functions obtained from maximizing the expected return of wealth measured in the currency basket with no constraints on portfolio rollovers or other constraints. The second column represents outcomes of the four densities when maximizing the same preference function on wealth again measured in the currency basket, but now conditional on a value-at-risk constraint on the downside risk in terms of dollar wealth, a constraint on negative returns measured also in dollars, and with some limits on the allowable maturity structure of the portfolio.

As can be seen, the "reshaped" density functions have probability masses for wealth that are much narrower than the original, unconstrained solutions. The tools by which the densities are reshaped include preference functions, CVaR constraints, and general constraints. The tables at the bottom of each column summarize the expected values of the aggregated portfolios in each case. This example produced a final solution with little risk measured in terms of the currency basket. The actual portfolio solution was a mix of 80%, 15%, 5% US dollars, Euro and yen respectively, which was close to the currency basket chosen (which itself represented the proportion of imports of the country in each currency). Interesting, and in hindsight, the long-duration portfolio as determined here turned out to be very desirable for the country's overall ALM.

4.3 Examples of specific central banks

We have discussed most of the generic elements of foreign exchange reserves risk management. We show here that these individual elements typically apply to many countries, but that these elements are not necessarily currently fully integrated. For some examples of actual

[14] RisKontroller™ is the software system from The RisKontrol Group implementing the framework discussed in this paper. It is called *"Shape the Future"*™ technology.

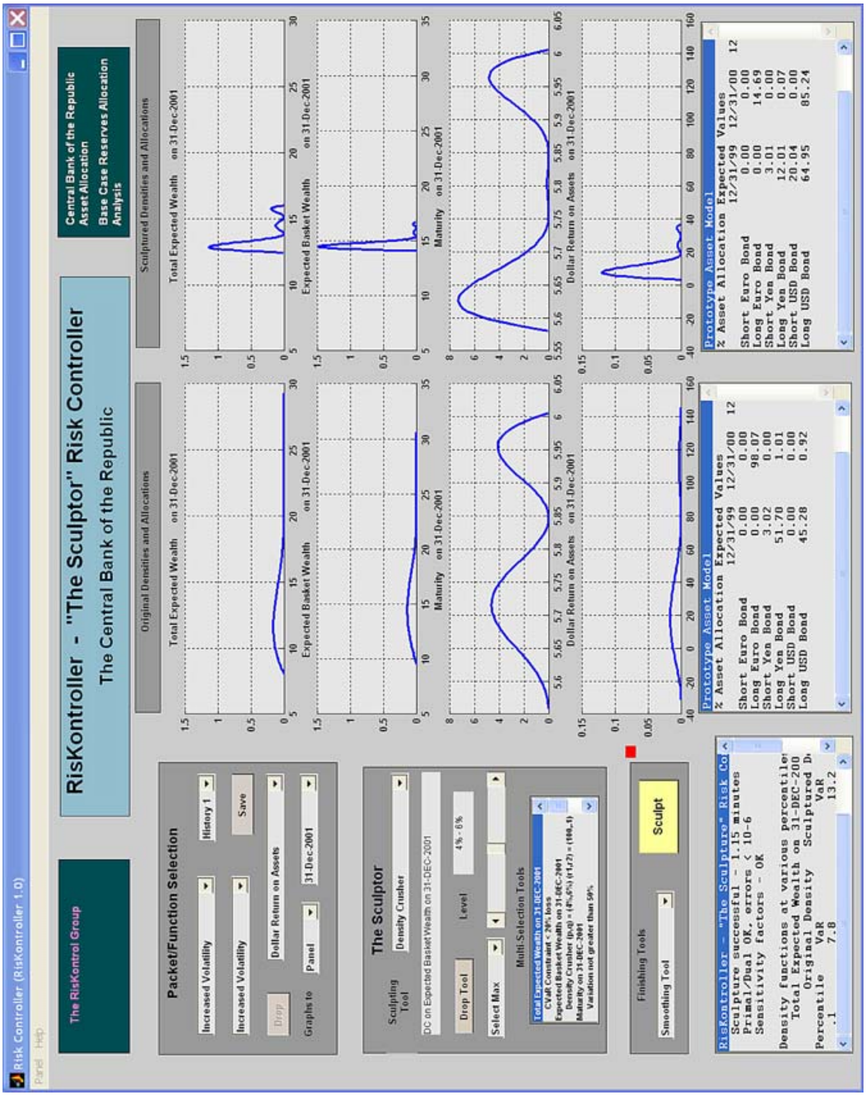

Fig. 3 Density shaping example

foreign exchange reserves risk management, we rely, besides on our own experiences, on IMF (2003)[15] and ECB (2004). We argue for each example how our approach can improve on existing reserves management approaches.

Most all countries surveyed in IMF (2003) have defined the strategic risk-return tradeoff in the form of a benchmark but most compute the benchmark in an ad hoc way. For example, they will compute currency allocations and asset allocations separately in order to determine benchmarks for currency risk and interest rate risk. Many countries use value-at-risk, VaR, or liquidity-at-risk, LaR, but they use these as a measurement tools and not as part of constraints within an overall model in order to obtain benchmarks. Some countries will use optimization models but these are mostly mean-variance models and do not contain multiple factors over multiple time periods incorporating multiple risk constraints as is done here. A number of countries simply match assets and liabilities but this exercise fails to account for the joint effects of currency and exchange rate fluctuations coupled with liquidity requirements in a dynamic framework. A further challenge confronting reserve managers is to avoid an actual negative return. This issue is handled in our framework, directly as in Eq. (18) or with a small probability of occurrence as in Eq. (36), by integrating this requirement with all the other risk constraints and other policy issues.

One example of typical reserves risk management approach is that used in Canada, Hungary, and Turkey. Here, the strategy first seeks to match currencies of assets and liabilities and then to select specific portfolio tranches, considering liquidity and other objectives and risk constraints. The managers do so in a piecemeal fashion, however, which is unnecessarily constrained and likely less than optimal. In our framework, currencies of assets and liabilities are handled simultaneously with liquidity and other risk factors. It leads to more optimal results, rather than unnecessarily constrained solutions.

Another example of sub-optimal management is that of Australia, Brazil, Chile, Colombia, Hong Kong SAR, Korea, Mexico, New Zealand, and the UK that all use VaR to monitor or limit market risk. They use VaR, however, primarily as a measurement tool and not as a management tool. In our case, we use CVaR constraints, and therefore limits on VaR, for several risk factors simultaneously to produce a benchmark satisfying the required constraints. Of course, we also get the desired VaR-quantities for risk measurement purposes, but that is secondary to deriving an optimal solution and a strategic benchmark.

Another example is that of Australia, Brazil, Colombia, and the Czech Republic, which all use basic techniques to minimize the probability of a capital loss over a certain time period. They will do this by setting duration limits and defining a VaR-type constraint. These countries do so, however through iterative simulations and expert judgment. With constraints such as (18), we can control the realized and unrealized losses with respect to specified limits (along with other risk factors) ex-ante and simultaneously, thereby deriving optimal solutions, rather than iterative solutions.

While Australia, Hong Kong SAR, Mexico, Norway, the UK, and New Zealand all use optimization models to produce long-term risk profiles of currency, risk, and return considerations, they do these for separate risks, do not combine multiple risk constraints as we have done here, and do not use a dynamic framework as we have done. None of these combine all the factors at the same time such as discussed in Section 3.

Another example of sub-optimal management is that many countries, including Chile, India, Israel, Korea, and Hungary, use a multitude of factors in their foreign exchange

[15] This paper is a supplement to the IMF's Guidelines for Foreign Exchange Reserve Management. It provides case studies of 20 countries and an overview of the techniques they use for reserves management and risk management of the reserves portfolio.

reserves risk management, but not in an integrated fashion. They consider factors such as the composition of external debt, particularly sovereign debt, the composition of trade, domestic currency basket, the intervention currency, and certain financial considerations. However, they typically do so in a piecemeal fashion. For example, India uses trade-based, money-based, and debt-based indicators as measures for reserve adequacy and separately obtains benchmarks for currencies, instruments, and duration. The model in 3.2 could be applicable in these cases as it integrates all these factors and risks at the same time.

Our framework can also be used to manage and assess reserves adequacy. One way of doing this is to use the indicators as suggested in de Beaufort and Kapteyn (2001). They consider one of the most important indicators for determining reserve adequacy in emerging market economies as the ratio of reserves to short-term-debt. Combining this ratio with other factors they determine an indicator and an interval over which that indicator can vary for adequate reserves. The model in Section 3.1 could be used to obtain strategic benchmarks for managing their indicator in the sense that optimal trajectories over time could be obtained in order to most likely fall within the indicator range. We can then produce benchmarks and "optimal" levels of reserves in the same model.

5 Conclusions

We have presented a framework for foreign exchange reserves management that combines tactical asset allocation considerations with broad macroeconomic, macro-prudential risk, and sovereign debt management considerations. Our foreign exchange reserves management framework allows for very general objective definitions and does not restrict the class of eligible stochastic processes or limit the investment universe. It also allows for easy feedback between outcomes and decision variables by including various tools that can reshape densities. The model can also be operated using a PC-based platform. We show the possible applications of our approach to several common reserves management problems, how it could be used by central banks, and the advantages over alternative approaches.

We see our approach as an important complement to the various ALM-tools currently being used by central banks and ministries of finance (and commercial banks) around the world. Our approach is more demanding than other approaches in terms of analytical modeling regarding objectives, constraints and assumptions. We think this is worthwhile for several reasons. Strategic allocation and the formation of benchmarks can be one of the most important determinants of portfolio returns and overall reserve management objectives. It is therefore important that the strategic analysis be done incorporating all the risks and other factors in an integrated fashion. Second, a dynamic framework that allows for rebalancing is very important to benchmark creation. It allows for incorporating longer-term effects such as mean-reversion of exchange rates, changing correlations, extreme events and contingent liabilities and the impacts of instruments such as longer duration assets and derivatives. Next our objectives and the risk constraints that we introduced allow us to capture multiple central bank considerations for safety, liquidity, stability, and returns. And it allows us to consider models incorporating not only assets and liabilities, but all kinds of risk indicators and objectives that are important to central banks. Lastly, the existing approaches used by central banks do not provide for the richness of the kinds of issues that can be addressed in stress tests as our framework can.

Our approach has also important advantages because it requires a very explicit process of model development that will help clarify the strategic aspects involved in risk management.

Often, these strategic aspects, including linkages between macro-, micro-economic and financial risks are left out or treated only implicitly in other approaches. At the same time, the use of numerical approaches to solving using a dynamic stochastic framework allows for analytical rigor while maintaining ample modeling flexibility.

Finally, as implemented, by using density and density shaping as major tools, the model retains many intuitive features attractive to policy makers. The approach not only provides the complete density functions of outcomes of a selected portfolio (instead of just summary measures), but also defines criteria for the shape of density functions in terms senior management can understand. Then, satisfying those criteria, our approach determines the strategic portfolio allocation that is optimal over time with respect to an objective—such as returns or the likelihood of an indicator, such as reserves to short-term debt, falling within a specified range. This intuitive approach can be supported at each level in an institution, while its flexibility allows adaptation to the unique requirements of each individual central bank. Because of its extremely flexible structure, the framework is not only applicable to developed countries but also to developing countries that face greater economic and institutional constraints.

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