Strategic Investment and Risk Management

for Sovereign Wealth Funds[1]

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1. Introduction

1.1. The case for analyzing SWFs' ALM

Sovereign wealth funds (SWFs)[4] have been around for a long time, at least since the 1950s. But their total size worldwide has grown dramatically over the last few years. Buoyed by rising commodity prices (at least until recently), desires to build up reserves for risk management purposes, and large capital inflows, many emerging markets as well as some developed countries have accumulated large SWFs in the last few years. The IMF estimated that assets of SWFs will rise from $2-3 trillion last year to about $6-10 trillion within five years (IMF, 2007). Other estimates are that these SWFs can grow to $11-14 trillion by 2012 (McKinsey, 2007; note oil and many other commodity prices have been very volatile lately, so predictions as to future size are fraught with large uncertainties). At present, China, Kuwait, Norway,

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[1] The views expressed in this paper are those of the authors and do not necessarily reflect the views of the IMF or IMF policy.

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[4] There is no unique definition of SWFs, but generally these are thought to include assets held by governments in another country's currency, i.e., held in foreign exchange. They are managed using a continuum of institutional mechanisms (Truman, 2008). Toward one end of the spectrum are pure SWFs, which generally have long-term investment objectives. These may include national pension, endowment and development funds. Further along the continuum are stabilization funds accumulated from "excess" revenues from commodity exports in particular. At the other end are traditional international reserves managed by central banks and/or finance ministries, where shorter-term considerations of liquidity and low risk normally are paramount. When purely held for exchange rate management and monetary policy purposes, however, these foreign reserve positions are typically not considered SWFs, although there is no sharp dividing line.

Russia, Saudi Arabia, Singapore, and the United Arab Emirates are among the countries that have the largest SWFs.

This rise of sovereign wealth assets is a new phenomenon with major implications for global capital flows, cross-border investments and international capital markets. Many international efforts are underway to analyze various aspects of SWFs. These include the desired governance and accountability arrangements for SWFs, whether there is a need (and if so in what forms) for oversight of their investments, e.g., in strategic industries, and how to mitigate concerns about a possible use of SWFs as foreign policy tools. Sets of best practices and guidelines are being developed on some of these issues (by the IMF, OECD).

These efforts do not address, however, the question as to how an SWF should manage its assets given its objectives, constraints and (possibly) guidelines, and specifically what constitutes its best asset mix. We will analyze this question from the perspective of an SWF seeking to meet its objectives of maximizing return (or wealth) given its risk management concerns and various constraints. We will first present a very general approach, yet one that can be easily adapted since each SWF faces specific objectives and constraints, making for unique ALM problems. We show how our framework avoids many of the shortcomings of other, commercial risk management systems. To show the specific benefits of our approach, we will apply our framework to the case of Norway's SWF, the Government Pension Fund - Global (NGPF or the *Fund*) of Norges Bank. Using the model, we will develop optimal benchmark asset compositions, compare our asset composition with their actual allocation (in terms of the benchmarks used), and discuss the results.[5] We end the paper with showing how our framework is applicable to a broad class of funds, including stabilization, national pension, endowment and development funds[6].

1.2 Why ALM for sovereigns and SWF s differ from the general ALM case and what our approach entails

ALM for a sovereign is not just a special case of the ALM-models developed for financial institutions, such as commercial banks or pension funds. Sovereign ALM has a number of special considerations. This importantly includes much longer horizons, as the ALM needs to address issues 10 to 30 years out, as opposed to, say, the day-by-day risk management of a commercial bank. Different objective functions will typically be involved, such as maximizing overall country's welfare, as opposed to, say, short-term

[5] The model only used historical data up until September 5, 2008 and did not use any additional data or expert views since. This way we make a fairer comparison with the benchmark we assumed for NGPF as we avoid the period of extreme turmoil during the fall of 2008.

[6] Analysis on the NGPF results that couldn't be included here because of space is available on the web site www.RisKontroller.com, which we subsequently refer to as the web-paper.

profitability. And different constraints will come into play, such as the need to observe legal constraints and report to parliament, as opposed to, say, satisfying shareholders' interests. These considerations already imply that standard ALM approaches do not suffice for the general case of ALM for sovereigns.

Yet, while a more strategic approach to risk management for sovereigns is needed, it is often not considered (Claessens, 2006). This is in part the case because commercial providers tend not to provide the right type of tools. This is because it is difficult and expensive to manage and maintain customized models in a dynamic stochastic optimization (also called dynamic stochastic programming) framework (see Kreuser, 2002 for various approaches to manage these models). Furthermore, there is little demand from the users' side for more sophisticated tools as the benefits of using them are often not well known and in any case difficult to explain to the principals overseeing sovereign risk managers (which can vary from Ministers to members of parliament and the general public). As such, the tendency for sovereign risk managers is to go along with classical risk management solutions, although their needs are different from most commercial systems.

Our approach addresses many of these shortcomings. As in other ALM approaches, we incorporate assets and liabilities to create balance sheets by currency at any point in time, allow for rebalancing of the portfolio and transfers between different currencies, and include transaction costs and various kinds of stochastic flows. We deviate, however, from the standard assumptions often used in financial models, such as the reduction of the problem to a mean-variance tradeoff and lognormal distributions. Different from many models, the scenario procedure we use generates trees so that at each point in time the model cannot anticipate what the future outcome will be as there will be several possibilities similar to the way uncertainty unfolds in reality. We also incorporate capabilities, more than usual, in the processes underlying the scenario tree generation. Specifically, we can allow the user choices of theories such as mean reversion in asset prices, Uncovered Interest Parity (UIP), expert judgment, historical estimation, or estimation of price processes from current instrument values. We can also incorporate changes in volatility and correlations depending on the level of state variables and time, the trending of currency and interest rates, the presence of fat tails in assets prices or the occurrence of extreme events. We can also account for broader changes in the international financial and economic environments and the occurrence of contingent liabilities. We then optimize the portfolio in an appropriate objective over the entire scenario tree subject to various policy and other constraints, while incorporating several risk constraints, possibly simultaneously at several points in time and at different points along the distributions of outcomes.

The very general problem we thus set up belongs to the class of dynamic stochastic optimization models which has proven to be very powerful for risk management problems. There is a literature that describes these models, either in their academic or specialized applications to central banks or ministries of finance and other institutions (Claessens and Kreuser, 2007 and other papers in the same volume; Zenios and Ziemba, 2007; Ziemba, 2003; Ziemba and Mulvey, 1998). Ziemba and Ziemba, 2007 and Mulvey et. al., 2002 describe the benefits of this approach nicely, and we do not expand on that here. We do want to point out though, that these approaches deviate from other methods of risk management.

Mean-variance approaches are often used by risk managers, but these models are very limited as they reduce the dynamics to the recurrence of a one-period problem. Multi-period models have significant advantages over single-period models, especially for long-term investors (Mulvey, 2002). At the same time, mean-variance approaches are sensitive to input assumptions, especially regarding expected returns. Often these models require limits on individual assets to improve diversity. The multi-period model will generally make more sensible recommendations, with more diversity. It will typically not put 100% in one asset because it hedges against all defined scenarios. Mean-variance also does not allow for periodic goals or interim constraints. In contrast, multi-period models have less need for such constraints. Single period models can not consider transaction costs in a natural way. And they do not take advantage of gains from rebalancing portfolios to target allocations (it has been shown that higher returns can be achieved by periodic rebalancing).

Sequentially applying mean-variance over time has also been proposed. But it has been shown that this approach is not optimal in the presence of transaction costs, future liquidity constraints, changing economic and financial conditions, and future binding risk or policy constraints. All factors that SWFs need to take into account.

Another version of the stochastic optimization approach is called the Fixed-Mix approach. This generates scenarios but not a tree. It does incorporate rebalancing at future dates but now the model has perfect information on what the future looks like after the initial uncertainty has been resolved. The way the approach then works is to select a fixed asset allocation over all possible rebalancing dates over all scenarios. This approach is often used because of the appeal of using scenarios which are easier to generate and conceptualize than trees. However, it does not consider changing the asset allocation over time, no matter how good or bad the scenarios have become, which is not realistic. Furthermore there is extensive evidence (e.g., Ziemba and Ziemba, 2007) that the performance of dynamic stochastic optimization dominates the Fixed-Mix approach. Another approach is called the Contingent Claims Analysis (described by Gray, 2007). This approach relies much more on asset price data to infer risk exposures and looks at the country balance sheet, but has less applicability to SWFs as it is not an optimization approach.

There are various kinds of closed-form solutions. One is given in Scherer, 2008 that obtains a solution using dynamic programming. These models suffer primarily from the inability to incorporate inequalities. The method also assumes normality and cannot include changes in economic and financial regimes. They also usually require an estimation of utility. Another related method is given in Tanaka, 2008 that suffers from the same difficulties. However, they can be used to get broad directional outlines. Both of these papers are applied to the NGPF and both obtain broad directional movements similar to what we obtain in the more detailed solutions using our framework.

Overall, and especially since long-term investors like SWFs have to satisfy several objectives beyond just risk and reward, the dynamic stochastic optimization approach combined with goals, objectives, and multiple risk constraints is the best way to go.

1.3 Why our approach is especially applicable to SWFs

Many SWF situations represent large deviations from the world of a financial institution's ALM. And every SWF faces a special form of ALM given its particular objectives, multiple risk constraints, dynamic rebalancing, changing stochastics over time, extreme events, and often very diverse classes of assets and liabilities to be managed. This requires not one specific model but rather a framework to facilitate building models unique to each SWF, which is what our approach allows. Since it is very flexible and can be adapted to sovereigns who face a multitude of risks and objectives, many of which are not easily captured using the standard models, our approach will fit a variety of SWFs' needs.

We can show this with some examples. For one, the function to be optimized can be adjusted depending upon the objectives of the SWF. Some SWFs, for example, stabilization funds, behave like and have objectives somewhat similar to central bank reserves. Central banks generally manage their reserves with the objectives of safety, liquidity, returns, and stability; generally in that order of importance. By stability it is meant the orderly transition of indicators over time. For example, it is generally preferred that no large jumps occur in the shortfall in the liquidity portfolio over time. While modest amounts of shortfall may be tolerable, large amounts are usually not. Our framework can handle these problems, even when there are multiple kinds of risks to control for and when extreme events need to be taken into account. Furthermore, smoothing cash flows such as to the government budget can be easily handled in this framework.

Other SWFs may be similar to, and in fact are, national pension funds that (also) invest in assets in foreign exchange. Pension funds may have as their primary objective having sufficient liquidity to make any interim payments on their liabilities while investing safely and maximizing long-term wealth. As for stability, they will want their funding ratio to be relatively stable over time so that there are no or very few jumps in contribution requirements. Yet investing in very safe assets will be costly, as the pension funds need to make high returns over the long-term. Endowment funds on the other hand will want to maximize long-term returns but because they will also have risky asset classes, future multiple risk constraints at different points in time and at different points along the distribution will be essential.

The other feature that makes our model more consistent with SWFs' needs is the ability to include macro-economic and micro-based concerns and constraints on sovereign wealth management. Macro-economic concerns can include issues arising from a country's currency management, debt management, and domestic financial sector stability. Micro-economic concerns can arise from legal or disclosure constraints, or be due to institutional capacity constraints. The SWF portfolio manager will have to account for these macro- and micro-economic factors. Specifically, it means commodity price behavior, resource depletion, import and export levels, fiscal issues, GDP, population growth/decrease, benchmarks, asset universe, or other factors may need to be included. And, as a consequence, each fund will be considering differing liability profiles, different instruments, and different investment horizons. Our framework is designed to take all these factors into account.

Lastly, our framework is easy to modify both in terms of the model and in terms of the scenario tree generation. It is entirely open as the model can be easily understood by non-experts and it is absolutely not a "black box". Information can be extracted from a solution that is suitable and tailored to any audience. The scenario tree generation is entirely independent of the model making it easy to generate different trees to apply to the same model. This is very important in testing sensitivity to tree generation and allows new kinds of stress tests where "stress" here means stressing the stochastics. This allows asking questions such as how to hedge against the probability of a "major" worldwide recession. We next present the framework, followed by the application to Norway and a discussion of its general applicability.

2. The general analytical framework

The point of departure for this paper is Claessens and Kreuser (2007). That paper described our general ALM framework for central banks and ministries of finance and combined assets and liabilities management. Building on that paper, this section describes the equations used to build models for various SWFs. For ease of exposition and application to Norway's SFW, NGPF, we will formulate the model as an asset only model, i.e., without explicit liabilities, but with exogenous fund inflows. We will select subsets of these equations for specific kinds of SWFs and for the NGPF, which we will describe in section 3.3. For the general details of scenario tree generation, we refer to Claessens and Kreuser (2007).

2.1 Notation, variables and general definitions

We first present the notation and variables we will use in the models. We adopt the following notation to characterize the topology of the tree.

$$X^t \equiv \{e \mid \text{event } e \text{ occurs in time period } t\} \tag{1}$$

$$AT^{t,e} \equiv \{(\tau,\varepsilon) \mid \text{For } \tau < t, \text{ event } \varepsilon \text{ preceeds event } e \text{ at time } \tau\} \tag{2}$$

The set definition (1) indicates what events (also read state) occur for each time period and (2) indicates the events that precede any event in any time period.[7] The sets that characterize the asset classes, the currencies, and time periods are:

$$\begin{aligned} I &\equiv \{\text{set of all asset classes, } i\} \\ C &\equiv \{\text{set of all currencies, } c\} \\ T &\equiv \{\text{set of all time periods, } t \text{ with } t = \bar{t} \text{ the last time period}\} \end{aligned} \tag{3}$$

As we will not use liabilities in the formulation of the NGPF in this paper, we omit the notation and equations for them. Decision variables are then defined for each event with respect to the levels of assets, currencies, alternative investments (derivatives), and other cash flows. Decision variables (including shorting of assets) are all handled separately, allowing the specification of transaction costs, spreads, and limits to be individually imposed and thus increasing the stability and realism of the model.

We define the variable $A_{i,c}^{t,e,\tau,\varepsilon}$ as the holdings of asset class i in currency c purchased at time t[8] and event e and held at time τ and event ε. We use the convention of capital letters for those variables whose values are the results of the dynamic stochastic optimization model, and smaller letters for all other variables (with the exception that capitals are also used for names of sets). Definitions of all the model variables, tree variables, and parameters are given in the Annex.

2.2 Extending the framework to SWFs

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[7] This mathematical notation can be directly translated into the General Algebraic Modeling system (GAMS) notation for optimization.

[8] We find this notation convenient especially for asset classes most often held by sovereign institutions. The notation is different than is normally used as we measure the purchase value while others often measure the number or volume of assets. There are reasons to use either notation for specific problems.

One typical special characteristic of SWFs is the focus on long-term wealth preservation and maximization. Similar to an individual aiming to maximize wealth for its retirement, many SWFs aim to generate the highest wealth for the nation and its citizens. Whether it has specific liabilities, as might be the case for a fund needing to meet the formally defined pension obligations to (future and current) generations, or whether it has a more loosely defined obligation to its citizens, as might be the case for some of the more recently established funds, will vary by SWF. Whether formal or not, these differences will, of course, affect the ALM problem (in one case, the SWF will have to worry much more on downside risks; in the other case it can more easily maximize wealth with less concern for risk).

To proxy the objective of long-term wealth preservation, one might wish to maximize a log utility function (also called Kelly betting or Capital Growth Theory, Ziemba, 2007) as this will yield the highest long-run levels of expected wealth. However, this can also produce very high levels of volatility of wealth. In Claessens and Kreuser, 2007, we developed a preference function that has some similarities with a utility function but addresses the risk of volatile final wealth. It also requires no special parameter estimation or a priori definition of a user's utility function. We describe that function below for a maximization problem (a minimization problem is similar). The function is like a utility function[9] in that it has a positive first derivative and non-positive second derivative. Therefore in maximizing the function (4), we will be maximizing TW^t but it operates considerably different. The interval $[p,q]$ is the interval in which we would like to push as much of the probability mass of the density function as possible. The values r_1 and r_2 determine how strongly we wish to push the probability mass into that interval. The important aspect is that the parameters r_1, r_2 and $[p,q]$ can be defined explicitly and are not estimated.

Preference Function

$$\text{(4)} \qquad \text{MAX} \quad E\left[\sum_{t\in T}\delta^t r_1^t\left(q^t-p^t\right)\Theta_{\frac{r_1^t}{r_2^t}}\left(\frac{W^t-p^t}{q^t-p^t}\right)\right]$$

$$\text{(5)} \qquad \Theta_e(\lambda) = \begin{cases} \lambda & \text{for } \lambda \le 0 \\ \lambda - \dfrac{\lambda^2(e-1)}{2e} & \text{for } 0 \le \lambda \le 1 \\ \dfrac{\lambda}{e} + \dfrac{e-1}{2e} & \text{for } \lambda \ge 1 \end{cases}$$

[9] This is not the same as a quadratic downside risk utility function. For the later, see Infanger, "Dynamic Asset Allocation Strategies Using a Stochastic Dynamic Programming Approach", p-211, in Zenios and Ziemba, 2007, Vol.1.

This function is a piece-wise linear-quadratic and can be solved computationally very efficiently. This allows it to be used with problems that have a very large scenario tree and is well suited to SWFs that have wealth targets in years far off, such as might be the case for funds that behave like national pension funds. It is very computationally attractive since it is quadratic and linear.

A more general class of utility functions like $\Theta(x)=\frac{x^{\gamma}}{\gamma},\ \gamma\leq 1$ can also be used ($=\ln(x)$ when γ=0) but that requires obtaining ones risk parameter γ and the problems become more difficult to solve usually requiring a reduction in the size of the scenario tree. When the targeted rates of return are less well known, the SWF may wish to maximize discounted wealth over time. This is the case for endowment funds, for example, and the preference function then becomes:

(6) $$\text{MAX} \quad \sum_{t\in T}\delta^{t}W^{t}$$

When there are objective functions defined in terms of shortfalls below a specified target, as might be the case for a national pension fund, the following objective might be used.:

(7) $$\text{MAX} \quad \sum_{t\in T}\delta^{t}\left(W^{t}-\sum_{e\in X^{t}}\pi^{t,e}\Gamma\left(SHF^{t,e}\right)\right)$$

The variable $SHF^{t,e}$ represents the shortfall and Γ is a convex function. This approach is often used as a proxy for controlling risk, defined as the amount of the shortfall.[10] There are several problems with this approach in general; how to define the function Γ and how to combine several shortfalls in the same objective? Furthermore, this gives a nonlinear problem unless one uses a piecewise linear approximation. We will make only use of this function for Γ linear and instead control for risk of shortfalls using conditional value-at-risk constraints.

Another characteristic of a SWF can be its specific source of assets and/or (finite) income, and its associated objectives. Many SWFs are set up by oil exporting and other natural resource rich countries. Faced with a finite amount to exploit and export, the SWFs set up by some countries (such as Kuwait and Norway) aim to accumulate wealth so as to provide sufficient income for future generations. This differs from those countries such as Brazil and China whose SWFs have accumulated foreign assets due to large capital inflows or surpluses on their current account due to non- resources exports. These countries may see their SWFs more as a tool to temporarily invest their surplus funds earning more attractive rates than

[10] See Ziemba and Mulvey, 1998 for several examples. Shortfall is also called “embarrassment” to reflect the embarrassment in not meeting targets.

from traditional foreign exchange reserves. For these SFWs, the objective of discounted expected wealth may be more relevant

These characteristics and considerations mean adaptations to the model, which is usually complicated. Fortunately, our framework is so general that in essence we simply need to adjust the objective function, constrain the equations for the balance sheet variables, or add legal and policy constraints, along with any constraints for handling various risks.

2.3 Structural equations for SWFs

These are the main sets of equations that define the model and include balance sheet and cash flow equations and policy and legal constraints. Each SWF will have its own policy and legal constraints but the main balance sheet and cash flow equations are similar with the exception that, depending on the fund, some variables may not appear. The main equation (8) defines the cash flows in each time period, for each event, and in each currency. It balances all cash flows by currency. Then given these cash flow equations, we can immediately construct the balance sheets at any time *t* and event *e* or the expected balance sheet at time *t*.

Cash Flows: By currency, time, and event

$$
\begin{aligned}
&\underbrace{\sum_{i \in I} A_{i,c}^{t,e,t,e}}_{[\text{new assets}]} + \underbrace{\sum_{i \in I} tca_i A_{i,c}^{t,e,t,e}}_{\left[\substack{\text{transaction costs}\\ \text{on new assets}}\right]} + \underbrace{V_{c,d}^{t,e}}_{\left[\substack{\text{transfer of}\\ \text{currency } c \text{ to } d}\right]} + \underbrace{CASH_c^{t,e}}_{\left[\substack{\text{cash in}\\ \text{currency } c}\right]} \\
&= \underbrace{\sum_{i \in I} ina_{i,c} \langle \text{if } t{=}0 \rangle}_{[\text{initial portfolio}]} + \underbrace{\sum_{d \in C, d \neq C} \frac{tc_c \gamma_c^{t,e} V_{d,c}^{t,e}}{\gamma_d^{t,e}}}_{[\text{transfers of currency } d \text{ to } c]} \\
&+ \underbrace{\sum_{i \in I} \sum_{\substack{(\tau,\varepsilon) \in AT^{t,e} \\ \tau \geq t - mat_i}} \sum_{(t-1,f) \in AT^{t-1,f}} (1 + tcs_i) \eta_{i,c}^{\tau,t,e} \left(A_{i,c}^{\tau,\varepsilon,t-1,f} - A_{i,c}^{\tau,\varepsilon,t,e} \right)}_{[\text{asset sales}]} \\
&+ \underbrace{\sum_{(t-1,\varepsilon) \in AT^{t,e}} \left(1 + rc\right) CASH_c^{t-1,\varepsilon}}_{[\text{return on cash}]} + \underbrace{op^{t,e} stock^t \langle \text{if } c = \text{"}USD\text{"} \rangle}_{[\text{funds flows to NGPG}]} \\
&+ \underbrace{\sum_{(t-1,\varepsilon) \in AT^{t,e}} \eta_{\text{"}swaps\text{","}USD\text{"}}^{t-1,\varepsilon} SWAPS_c^{t-1,\varepsilon} \langle \text{if } c = \text{"}USD\text{"} \rangle}_{[\text{oil swaps flows}]}
\end{aligned}
$$

(8) $$\forall\ c \in C,\ t \in T,\ e \in X^t$$

The next equation ensures that the sales of assets are properly defined in the model, i.e. to be monotonic.

Monotone Asset Sales

(9) $$\sum_{(t-1,f)\in AT^{t,e}} \left(A_{i,c}^{\tau,\varepsilon,t-1,f} - A_{i,c}^{\tau,\varepsilon,t,e} \right) \quad \geq \quad 0$$

[monotonicity constraint so that sales are reflected in changes in holdings]

$$\forall\ i,c,t,e,\tau,\varepsilon \ \ni\ (\tau,\varepsilon)\in AT^{t,e} \text{ and } (t-\tau) < mat_i$$

Total Wealth: In numeraire currency.

(10) $$TW^{t,e} = \sum_{i\in I,c\in C} \frac{A_{i,c}^{t,e,t,e}}{\gamma_c^{t,e}} + \sum_{i\in I,c\in C}\ \sum_{(\tau,\varepsilon)\in AT^{t,e}} \frac{\eta_{i,c}^{\tau,t,e} A_{i,c}^{\tau,\varepsilon,t,e}}{\gamma_c^{t,e}}$$

[total assets] [new assets] [assets marked to market]

$$+ \sum_{c\in C} \frac{CASH_c^{t,e}}{\gamma_c^{t,e}}$$

[cash] $\forall\ t,e \ \ni\ e\in X^t$

The next constraint begins the definition of the preference function (4). It allocates value to the three segmentations of the preference function according to (12) so that they are assured their proper values due to the concavity of the function under maximization defined at the horizon $t=\bar{t}$.

Segmentation Definition 1(the following example is for the ratio of reserves to short-term-debt)

(11) $$\frac{TW^{t,e}}{std^{t,e}} - p^t = \left(q^t - p^t\right)\left(SG_1^{t,e} + SG_2^{t,e} + SG_3^{t,e}\right) \qquad \forall\ t=\bar{t}, e\in X^{\bar{t}}.$$

The next set of constraints ensures that the segmentations take their proper values.

Segmentation Definition 2

(12) $$SG_1^{t,e} \le 0, \quad 0 \le SG_2^{t,e} \le 1, \quad 0 \le SG_3^{t,e} \qquad \forall\ t=\bar{t}, e\in X^{\bar{t}}$$

Preference Function definition: Defined for $t=\bar{t}$ *and therefore set* $\delta^{\bar{t}} = 1$.

(13) $$PRF = \sum_{t=\bar{t},e\in X^{\bar{t}}} \pi^{t,e}\delta^t r_1^t \left(q^t - p^t\right)\left(SG_1^{t,e} + SG_2^{t,e} - (SG_2^{t,e})^2 \left(\frac{r_1^t}{r_2^t} - 1\right)\frac{r_2^t}{2r_1^t} + SG_3^{t,e}\frac{r_2^t}{r_1^t} \right)$$

The following defines the total expected value of the portfolio as measured in the numeraire.

Total Expected Wealth Definition

(14) $$W^t = \sum_{e \in X^t} \pi^{t,e} TW^{t,e} \qquad \forall\ t \in T$$

There are three separate definitions of shortfall that we use.

Shortfall Target: As might be used in targeting a funding ratio[11] *for a national pension fund.*

(15) $$SHF^{t,e} \geq ratio \times target^t - TW^{t,e} \ ,\ \ SHF^{t,e} \geq 0, \qquad \forall t, e \ni e \in X^t$$

Shortfall in Wealth: shortfall below expected wealth when targets are not known.

(16) $$SHF^{t,e} \geq ratio \times W^t - TW^{t,e} \ ,\ \ SHF^{t,e} \geq 0, \quad \forall t, e \ni e \in X^t$$

Shortfall in Liquidity: Shortfall in meeting liquidity targets.

(17) $$SHF^{t,e} \geq \sum_c \frac{liqreq_c - LQ_c^{t,e}}{\gamma_c^{t,e}} \ ,\ \ SHF^{t,e} \geq 0, \quad \forall t, e \ni e \in X^t$$

The parameter *ratio* in equations (15) and (16) is used to define the percentage of the targeted value that is desired. Since shortfall appears in these three equations as an inequality, it must be part of the objective function or part of a binding constraint. We will use specific constraints to control the risk in shortfall and will incorporate it in the binding constraint (see next section).

Asset Sales: By individual asset in time *t* and event *e*.

(18) $$SL_{i,c}^{t,e} = \sum_{\substack{(\tau,\varepsilon) \in AT^{t,e} \\ \tau \geq t - mat_i}} \ \sum_{(t-1,f) \in AT^{t-1,f}} \eta_{i,c}^{\tau,t,e} \left(A_{i,c}^{\tau,\varepsilon,t-1,f} - A_{i,c}^{\tau,\varepsilon,t,e} \right)$$

2.4 Policy constraints for SWFs

The policy constraints that might be imposed can be many and include limits on asset classes; portfolio rollover constraints; transaction cost limits; cash flow requirements; currency transfer constraints; market access constraints; liquidity constraints; etc. The following may be a desirable set of policy constraints:

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[11] The target for a pension fund could be the net present value of future liabilities.

Minimum Activity by Currency:

$$\underset{\text{[minimum activity as \% of total]}}{ma_c^t TW^{t,e}\gamma_c^{t,e}} \leq \underset{\text{[cash]}}{CASH_c^{t,e}}$$

$$(19) \qquad + \underset{\text{[new assets]}}{\sum_{i\in I} A_{i,c}^{t,e,t,e}} + \underset{\text{[assets marked to market]}}{\sum_{i\in I}\sum_{(\tau,\varepsilon)\in AT^{t,e}} \eta_{i,c}^{\tau,t,e} A_{i,c}^{\tau,\varepsilon,t,e}}$$

$$\forall\; c,t,e \;\ni\; e\in X^t$$

Minimum Allowable Return:

$$(20) \qquad nret^t \sum_{(t-1,\varepsilon)\in AT^{t,e}} TW^{t-1,\varepsilon} \;\leq\; TW^{t,e} - \sum_{(t-1,\varepsilon)\in AT^{t,e}} TW^{t-1,\varepsilon} \qquad \forall\; t,e \;\ni\; e\in X^t.$$

Maximum Portfolio Rollover:

$$(21) \qquad \sum_{(i,c)} \frac{SL_{i,c}^{t,e}}{\gamma_c^{t,e}} \leq percentrollover^t TW^{t,e}$$

Market Activity Limits:

$$(22) \qquad \underset{\text{[new assets]}}{A_{i,c}^{t,e,t,e}} + \underset{\text{[assets marked to market]}}{\sum_{(\tau,\varepsilon)\in AT^{t,e}} \eta_{i,c}^{\tau,t,e} A_{i,c}^{\tau,\varepsilon,t,e}} \leq upperlimit_{i,c} TW^{t,e}$$

2.5 Risks and risk control for SWFs

An important aspect of SWF management is the control of risk or, put differently, restrictions on the desirable distribution of final wealth, rate of return, or other variables one might wish to restrict, such as shortfall. The measure we use for controlling risks or shaping distributions (or densities) is Conditional Value-at-Risk (CVaR). CVaR is defined as the expected shortfall of a target at a specified probability or confidence level of α%. It is equivalent to the concept of the average loss (gain) in the worst (best) α% cases. CVaR is related to value-at-risk (VaR) and for loss distributions, CVaR is always greater than or equal to VaR. CVaR is a more robust measure than VaR, since it provides the average loss for those α% of cases rather than the minimum loss. It therefore takes more fully into account when losses can be extreme (Rockafellar and Uryasev, 2002, discuss CVaR in detail and Acerbi and Tasche, 2002, discuss the relationship between CVaR, expected shortfall, worse conditional expectation, tail conditional expectation, and VaR).

CVaR has some very useful properties for modeling purposes. Specifically, CVaR constraints can be modeled as linear constraints whereas VaR constraints are non-convex and non-smooth and thus more complex to model and solve besides the other problems that VaR has. This makes CVaR constraints the method of choice for shaping the distributions or densities for dynamic stochastic optimization models, especially when the models are very large.

CVaR has another important property useful for SWFs: it is a coherent risk measure (McNeil, Frey and Embrechts, 2005). The property of coherence important here is its subadditivity that makes decentralization of risk management systems possible. In particular, the CVaR risk of the overall loss is less than the sum of the risks of the CVaR losses of the decentralized portfolios. This is very important for SWFs that most often have a substantial part of their portfolio managed externally with separately defined objectives and limits. There are some idealized cases where VaR is coherent as well, but most often VaR (and semi-variance) are not coherent risk measures.

CVaR Constraints can be defined by time period, activity, confidence level, and level value and multiple constraints can be included.

(23)

$$Z^e \geq \Lambda\left(X^e\right) - ALPHA,\ \ Z^e \geq 0, \quad \forall e \in X^t, \text{ and for some } t$$

and

$$\sum_{e \in X^t} \pi^{t,e} Z^e \leq \rho\left(clevel_\rho - ALPHA\right)$$

Where Λ is a convex loss function, Z^e and $ALPHA$ are variables, $1-\rho$ is the confidence level, and $clevel_\rho$ is the value based upon that confidence level. $ALPHA$ will take the value of VaR corresponding to ρ in an optimal solution. Several CVaR constraints can also be placed on the same variable X^e at different confidence levels thereby shaping the distribution. Several CVaR constraints can be placed at different times and on different variables.[12]

CVaR allows us to translate intuitive descriptions of constraints on distributions into mathematical descriptions for use in our modeling framework. One example of a CVaR constraint can be that in the 1% worst cases, average portfolio losses should not exceed 10% of the portfolio value. Another example

[12] Since one equation is required for each event *e* it would seem that the model could become very large and difficult to solve. This is not usually the case since we will use linear Λ and only a small number of these constraints are in fact active making the problem easier to handle than would otherwise be assumed.

could be that in the 5% worst cases the average shortfall in liquidity should not exceed 20%. We may constrain a distribution of returns to certain loss levels at the 90% confidence level, the 98% confidence level, and the 99.9% confidence level, for example.

A common constraint used in SWFs is on the safety of returns, i.e., protection against extreme losses.

Wealth at Risk: shortfall is defined here as in equation (16)

$$(24)\ \Lambda\left(X^{e}\right)=SHF^{t,e}_{\text{Shortfall in Wealth}},\quad \rho=1-\text{confidence level},\ \ clevel_{\rho}=\text{ required expected shortfall}$$

Target at Risk: Target and shortfall are defined as in equation (15) and in the case of a pension fund it is funding ratio at risk.

$$(25)\ \Lambda\left(X^{e}\right)=SHF^{t,e}_{\text{Shortfall in Target}},\quad \rho=1-\text{confidence level},\ \ clevel_{\rho}=\text{ required expected shortfall}$$

Liquidity at Risk: Shortfall is defined as in equation (17)

$$(26)\quad \Lambda\left(X^{e}\right)=SHF^{t,e}_{Liquidity},\quad \rho=1-\text{confidence level},\ \ clevel_{\rho}=\text{ required expected shortfall}$$

Level of Rebalancing at Risk: This controls the distribution of sales

$$(27)\quad \Lambda\left(X^{e}\right)=\sum_{(i,c)}\frac{SL^{t,e}_{i,c}}{\gamma^{t,e}_{c}},\quad \rho=1-\text{confidence level},\ \ clevel_{\rho}=\text{ required expected shortfall}$$

Unrealized Losses at Risk:

$$(28)\qquad \Lambda\left(X^{e}\right)=\sum_{(i,c)}\left(1-\eta^{t-1,t,e}_{i,c}\right)\frac{A^{t-1,\varepsilon,t,e}_{i,c}}{\gamma^{t,e}_{c}}\Bigg|_{\eta<1}$$

$$\rho=1-\text{confidence level},\ \ clevel_{\rho}=\text{ required expected shortfall}$$

3. Application to Government Pension Fund – Global of Norges Bank

We next apply our model to Norway's SWF, NGPF. We will describe its purpose, the current asset mix, and the outcome of our optimization model. The model results reported here are preliminary as a more detailed study exceeds the scope and limited length of this paper. A more detailed report appears in the web-paper.

3.1 Background

Institutional setup, management principles, transparency

The Government Pension Fund was formally established by the Norwegian Parliament by the Act of 20 December 2005, but had been in existence in other forms for quite some time. The Government Pension Fund has two parts: the Government Pension Fund – Global (previously the Government Petroleum Fund, established in 1990) and the Government Pension Fund – Norway (previously the National Insurance Fund, established in 1967), with the second one very small. The Ministry of Finance has delegated the operational management of the Government Pension Fund - Global (NGPF or the *Fund*) to the Norwegian central bank (Norges Bank), with the mandate stipulated in a regulation issued by the Ministry and a management agreement further defining the relationship. The Fund itself is managed by Norges Bank Investment Management (NBIM), a separate part of Norges Bank, Norway's Central Bank.[13]

The Fund's inflow consists of all state petroleum revenues, the net results of financial transactions related to petroleum activities, as well as the return on the Fund's investments.[14] The outflow from the Fund is the sum needed to cover the non-oil budget deficit, which the government aims to keep at less than 4% of the size of the Fund. The Fund's current size as of January 1 2008 was NOK 2019 billion (some USD 375 billion at the current exchange rate of 5.4 NOK/USD) and is expected to grow to NOK 4350 billion by 2015. The Government Pension Fund is 14 billion NOK.

[13] NBIM also manages, besides the Government Petroleum Insurance Fund (which is small, only NOK 16 billion), the major share of Norges Bank's foreign exchange reserves.

[14] According to the Government pension fund Act (No. 123 of 21 December 2005), the cash flow is the sum of 1. total tax revenues and royalty deriving from petroleum activities collected pursuant to Petroleum Taxation Act (no. 35 of 13 June 1975) and the Petroleum Activities Act (no. 72 of 29 November 1996), 2. revenues deriving from tax on CO2 emissions due to petroleum activities on the continental shelf, 3. revenues deriving from tax on NOx emissions due to petroleum activities on the continental shelf, 4. revenues deriving from the State's Direct Financial Interest in petroleum activities, defined as operating income and other income less operating expenses and other direct expenses, 5. central government revenues from net surplus agreements associated with certain production licenses, 6. dividends from Statoil ASA, 7. transfers from the Petroleum Insurance Fund, 8. central government revenues deriving from the removal or alternative use of installations on the continental shelf, 9. any government sale of stakes representing the State's Direct Financial Interest in petroleum activities, less 1. central government direct investments in petroleum activities, 2. central government expenses in connection with the Petroleum Insurance Fund, 3. central government expenses in connection with the removal or alternative use of installations on the continental shelf, 4. any government purchase of stakes as part of the State's Direct Financial Interest in petroleum activities. Net financial transactions associated with petroleum activities are gross revenues from government sale of shares in Statoil ASA less 1. any government purchase of shares in Statoil ASA, defined as the market price paid by the government for the shares, 2. government capital contributions to Statoil ASA and companies attending to government interests in petroleum activities.

The Fund has no formally defined liability side and it is not specifically earmarked for pension expenditures, but it is the prime source for covering the government deficit. Its economic function can therefore best be compared to that of an endowment fund, similar to those that aim to be engaged in, say, philanthropy solely from the returns on its investments (Kjaer, 2007). The Fund's strategy requires that: "it shall be securely managed based on the objective of high return subject to moderate risk in order to contribute to safeguarding the basis of future welfare, including national pensions" (from the Fund's own Articles).

Asset and liability classes

The Fund can invest solely abroad and only in financial assets. Its management strategy is laid down using a benchmark portfolio consisting of equities and fixed income instruments. In 2006 changes were proposed to the benchmark portfolio, specifically to increase the equity allocation.[15] Current asset allocations are set at 60% equity and 40% fixed income. The equity portion of the benchmark consists of equities listed on stock exchanges in Europe (50 per cent), the Americas and Africa (35 per cent), and Asia and Oceania (15 per cent). The regional distribution (in terms of currency) of the fixed income benchmark is 60 per cent Europe, 35 per cent Americas and 5 per cent Asia and Oceania. Rates of return on asset classes are to be compared with those used to compute the benchmark for the Fund. These are Lehman FTSE All World indices for equities and Global indices for fixed income instruments for each of the markets/currencies USD, GBP, Euro, and Yen. The Norwegian Fund actual allocations are closely aligned with its declared benchmark allocations (Norges Bank Investment Management, 2008). We make the initial asset allocation in our model to correspond to the distribution of assets by region, fixed income, and estimated duration as given in NBIM, 2008. The details are given in Table 1.

Management, Follow-up and Control

Norges Bank manages assets largely internally (more than 75% of equity). NBIM hired its first external managers in January 1998, when the Government Pension Fund began investing in equity markets through index managers. The Fund reports results on a quarterly basis its rates of return in absolute terms and relative to its benchmarks. The auditing of the Fund has been assigned to the Office of the Auditor General, which bases its audit on the work performed by Central Bank Audit. The Fund is very transparent and information up to the level of investment in specific companies is available.

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[15] Notice from the Governor's Staff – Investment Strategy, Norges Bank, 2006b. In this change, the Fund's strategic asset allocation was compared with funds with similar objectives, two of which were the Harvard and Yale University endowment funds.

Table 1: Initial Asset Class Composition based on NBIM Benchmark on 5.9.2008			
Asset Classes	Percent Composition	Value in NOK (billions)	Value in local currency (billions)
Equity – FTSE AW Europe (Euro) Equity	14.00%	272.41	33.94
Equity – FTSE AW Americas (USD) Equity	14.00%	272.41	47.96
Equity – FTSE AW Asia Pacific (Yen) Equity	6.00%	116.75	2,215.77
Lehman Euro-Agg. 1-3 Year Bonds	15.75%	306.46	38.18
Lehman Euro-Agg. 10+ Year Bonds	5.25%	102.15	12.73
Lehman Global: US Treasury 1-3 Year Bonds	15.75%	306.46	53.96
Lehman US Agg. A+ 5+ Year Bonds	5.25%	102.15	17.99
Lehman Global: Japan 1-3 Year Bonds	6.75%	131.34	2,492.75
Lehman Global: Japan Long Bonds	2.25%	43.78	830.92
Lehman Global: UK 1-3 Year Bonds	11.25%	218.90	21.85
Lehman Global: UK Long Bonds	3.75%	72.97	7.28
Lehman Global: Inflation Linked Bonds	0.00%	0	0

Total portfolio value = NOK 1,945.80 billion. This amount is used for the model as the beginning date (September 5, 2008) value.

3.2 Income, foreign exchange risks and scenario generation

The Fund receives regular transfers from the Ministry of Finance, which in 2007 totaled some NOK 314 billion or on average more than USD 300 million per trading day. As noted, most of this income is from oil related revenues and thus volatile, as the last year has again made clear. Recent events have also shown that exchange rates can affect oil prices: as the dollar depreciated (appreciated), oil (and other commodity) prices denominated in dollar terms tended to increase (decrease). Theory and empirics also strongly suggest that there is also an inverse relationship between (real) interest rates and oil prices. And as oil is increasingly traded as an asset, we can expect oil prices to respond even more to the opportunity cost of capital. Given the importance of these relationships for the income stream of the Fund, we are careful on how oil prices are affected by foreign exchange and interest rates, and estimate these relationships, using Norway specific oil prices, with various econometrically robust techniques.

The foreign currency/Norwegian Kroner risk is not considered in the stated strategic ALM objectives of the Fund and, consistently, the Fund actually largely reports its performance and asset values in USD. This can be interpreted to mean that the Fund (implicitly) takes the view that its ALM should lead the Norwegian Kroner to be relatively stable or even increase in value, so as to maximize the purchasing power of the Norwegian population. Put differently, one of the main indirect objectives of the Fund will be to reduce Norwegian Kroner exchange rate risk so as to assure a stable income and consumption pattern. We therefore have to consider the consumption basket of the Norwegian population. The foreign exchange will be available to buy imports, but these imports can be in different currencies and coming from different markets. To mimic this consumption basket we define a currency unit, called the *NKU*, using the import shares of the four major currency groups, USD, GBP, Euro and Yen. Specifically, we classify imports from US, Latin America and China as "USD" imports; imports from the UK as "GBP" imports; imports from all of Europe outside the UK as "Euro" imports, and all imports from other countries as "Yen" imports. Using the trade amounts for the year 2006 as weights we then create the *NKU*, in which we then evaluate the risk and returns on the portfolio, i.e., we rebase all securities in the *NKU* today and at all future dates and scenario events. We take the NKU to be 69% Euro, 12% USD, 7% Yen, and 12% GBP.

One way we use to generate scenarios is to use multi-factor stochastic partial differential equations:

$$\frac{ds_i(t)}{s_i(t)} = \mu_i(s,t)dt + \sum_j b_{ij}(s,t)\sigma_j(s,t)d\omega_j(t) \tag{29}$$

This representation underlies commonly used multi-factor models, including those with mean-reversion, such as Hull-White and Heath, Jarrow, and Morton. The parameters (μ, b, σ) of these equations can be determined from historical data.[16] But they can also be adjusted for expert views, theories, and current implied prices.

We divide the interval of time from today, $t = t_0$ to the terminal period, $t = t_T$ into intervals $[t_0, t_1], [t_1, t_2], \ldots, [t_{T-1}, t_T]$ and use an estimate of μ, b, σ for each interval separately. This allows, for example, the correlations to depend on the time and the level of the variables, allowing the generation of a very rich possibility of outcomes. The solution[17] of these equations for each interval, in which μ, b, σ are constant is:

[16] Because of the way we structure our model formulation, we estimate most variables in the rate-of-return space. This makes the formulation particularly consistent and applicable especially in the formulation of the stochastic estimations.

[17] Solution provided by Roger J-B Wets, as in Claessens et. al., 1998.

(30)

$$s_i(t) = s_i^0 \exp\left[\left(\mu_i - \frac{1}{2}\sum_{j=1}^{n} b_{ij}^2 \sigma_j^2\right)t + \sum_{j=1}^{n} b_{ij}\sigma_j \omega_j(t)\right]$$

$$E\{s(t)\} = \left(s_1^0 e^{\mu_1 t}, s_2^0 e^{\mu_2 t}, .., s_n^0 e^{\mu_n t}\right)$$

$$\operatorname{cov}\left(s_l(t), s_k(t)\right) = E\{s_l(t)\}\, E\{s_k(t)\}\left(\exp(t\sum_{j=1}^{n} b_{lj} b_{kj} \sigma_j^2) - 1\right)$$

$$\operatorname{var}\left(s_l(t)\right) = \left(E\{s_l(t)\}\right)^2 \left(e^{t\sum_{j=1}^{n} b_{lj}^2 \sigma_j^2} - 1\right)$$

These equations provide the first and second moments for solving the moment matching problem. The scenario trees are generated by using the above expressions and obtaining moments such that the moment matching problem is minimized and that produces the greatest possible scenario spread. Scenario trees are based on the assumptions of the distributions chosen. The distributions are not normal because they are recalibrated at each rebalancing date and because of the way scenario trees are generated. The scenarios are generated so as to avoid arbitrage possibilities (Cornuejol and Tütüncü, 2007; it turns out that arbitrage is not a frequent problem).

The stochastic processes can be visually checked by looking at the scenarios generated. In the Figure 1 we see a small number of scenarios sampled. This diagram shows also the history used to estimate the processes. In this case, this process is assumed to be trend reverting. The stochastic process parameters are all estimated simultaneously, including the instantaneous correlations. The red line shows how the expected value process moves over time. The process incorporates short-term movements with long-term movements such as trend- or mean-reversion.

These processes are then converted into scenario trees[18] as in the Figure 2. The scenarios in the scenario tree are not of equal probability and this is why the scenario tree can be generated including extremes, even though there are not a large number of scenarios. Similar, but not identical, methods of generating scenario trees can be found in Zenios and Ziemba, 2007; especially in chapter 6 by Kouwenberg and Zenios.

[18] We avoid the “curse of dimensionality” by building good trees using moment matching, scenario reduction, and tree spreading. See Claessens and Kreuser, 2007.

Figure 1: Estimated stochastic process in the short- and long-term.

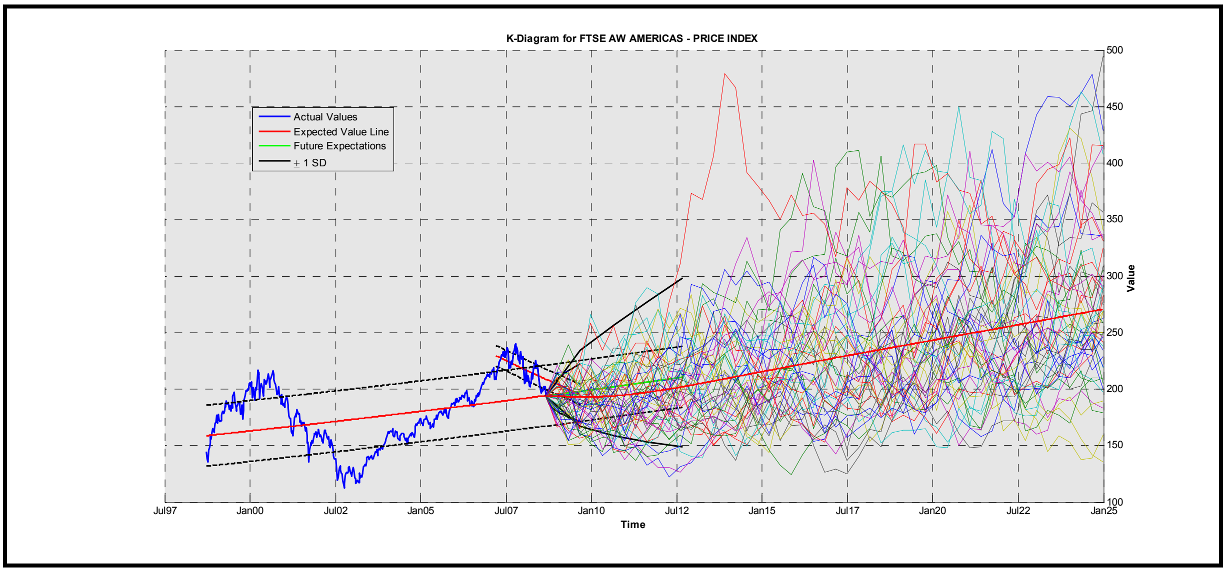
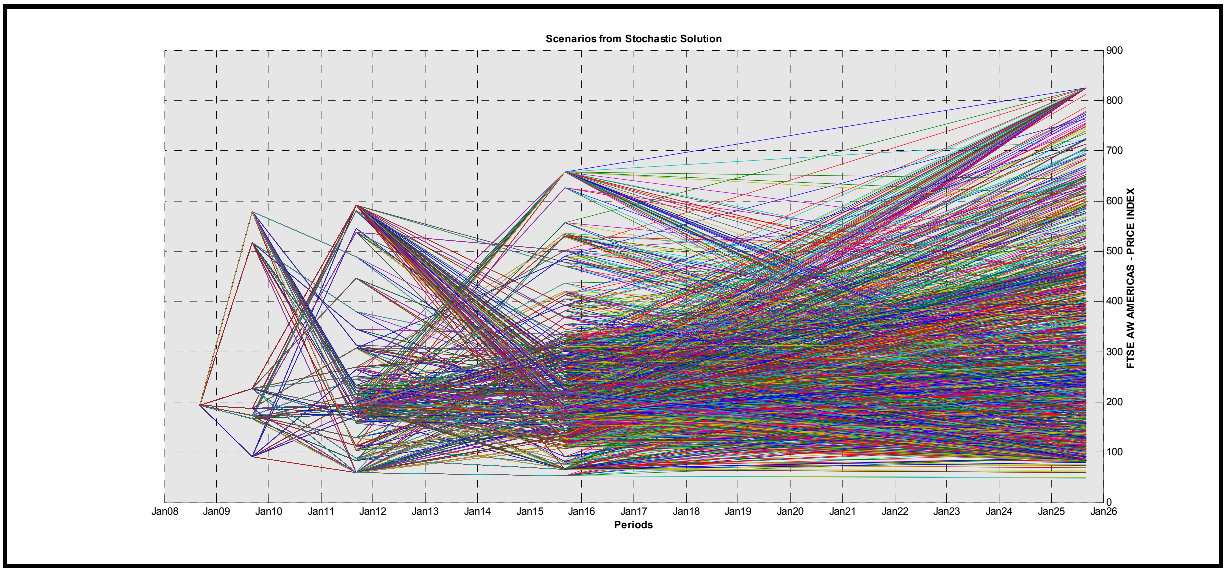


Figure 2: Tree created from stochastic processes



We provide in Table 2 some statistics for two processes; equity in US$ and the US$ long bond. The stochastics vary over time but not substantially. We could have imposed an expert view and increased the volatility over the short-term. The processes also tend to converge toward a normal process in the long-term. The annualized return on equity in US$ may seem a little low. However, if we look at the graph of the scenarios, which we have illustrated in Figure 1, they do not look unreasonable just using historical information. The NKU return tends to be higher than the US$ return and that is because the US$ is strengthening over the long-term. The Sharpe ratio as measured in NKU, and we are optimizing our model in NKU, is much greater for the long bond. The model solution will tend to favor the long bond over equity.

Table 2: Stochastic Process Statistics

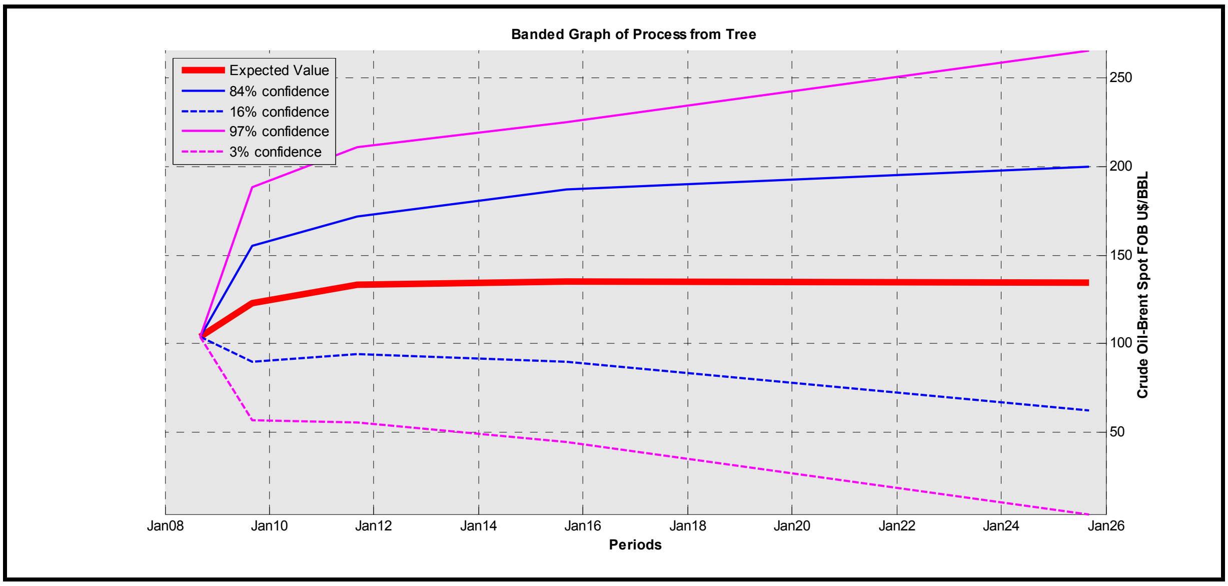
Stochastic Process	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Equity – FTSE AW Americas Price Index					
Mean Value in US$	193.38	193.04	197.25	219.17	272.91
Annualized US$ Volatility %		12.87	15.23	16.70	17.80
3rd Moment[19]		2.27	2.88	.56	.73
4th Moment[20]		8.72	8.64	1.86	1.06
Annualized US$ Return		-0.18	1.08	2.66	2.22
Annualized NKU Return		-4.12	3.54	5.53	2.88
Sharpe Ratio in NKU		-0.23	.28	.47	.36
Lehman US AGG A+ 5+ YR					
Mean Value in US$	188.23	195.76	222.76	294.54	548.14
Annualized US$ Volatility %		4.70	4.05	3.15	3.29
3rd Moment		25.16	8.33	3.95	1.40
4th Moment		337.67	53.18	17.47	6.50
Annualized US$ Return		4.00	6.67	7.23	6.40
Annualized NKU Return		-0.10	9.12	10.07	6.97
Sharpe Ratio in NKU		-0.01	1.50	2.18	2.14

We also estimate the oil price behavior and its correlations with the other asset classes. We use, however, a deterministic oil volume. We did not have enough information to estimate the behavior of the government deficit so we did not include that or budget transfers as a stochastic variable. Furthermore we could have included the extraction rate as a stochastic decision variable with bounds. This could be useful

[19] Scaled by dividing by the standard deviation squared.
[20] Scaled by dividing by three times the variance squared.

in arguing quotas for sovereigns. We show the graph of the distribution of the oil price[21] over time in Figure 3.

Figure 3: Oil Price Stochastics



3.3 Objective functions and policy constraints

The Fund's purpose can be stated as: maximize the expected end of period wealth in the currency basket *NKU*, given the uncertain and limited income stream of oil-related revenues, and observing risk constraints and policy constraints. This objective aims to preserve the income level of future generations when the oil revenues run out. After that the Fund is assume to have a payout ratio that will keep its overall real wealth constant over time. This ratio itself will have to vary over time as rates of return on the assets vary, but that problem is not analyzed here.[22]

The complete set of constraints includes:

- Maximize Expected Wealth (equation 6) measured in NKU.
- Cash Flows equation (equation 8) with swaps and funds flows based upon the oil price.

[21] Although this distribution missed the real situation, it is not unreasonable using only historical information. We could have imposed an expert view on this and assumed more volatility at the least but wanted our results to be more comparable with results obtained by other risk management methods.

[22] This is consistent with the current fiscal rule that stipulates that, over time, the central government non-oil structural deficit can not be larger than the real rate of return on the assets of the Fund, currently estimated at 4%, thus preserving the real value of the Fund.

- Monotone asset sales (equation 9) to define sales properly.
- Total assets definition (equation 10).
- Expected wealth definition (equation 14) defines wealth in USD.
- Shortfall in wealth definition (equation 16) to control downside risk in wealth.
- Minimum activity by currency (equation 19) to maintain a minimum amount in each currency. These values are 30% of wealth measured in NKU for Euro, 5% for USD, 3% for Yen, and 5% for GBP for each rebalancing period and scenario.
- Maximum portfolio rollover (equation 21) to prevent the entire portfolio from being rolled over. No more than 25% of the portfolio can be rolled over initially. After that as much as 100% of the portfolio can be rolled over at any one rebalancing date.
- Wealth at risk (equation 24) measured in NKU; this defines the CVaR constraints such that in every period the expected tail loss at the 95% confidence level should not exceed 20% of the expected portfolio value in NKU

Smoothing transfers to the government budget can be taken into account using a version of equation (25). This process is similar to that used in smoothing contributions to a defined benefit pension plan in these kinds of models.

3.4 Risks and risk control constraints

We put CVaR constraints on the downside wealth at each rebalancing point. We use separate constraints at the final date, when the oil runs out, and at the interim dates. At the final date a constraint is necessary since the Fund can not afford to have its assets fall below a certain level such that it can no longer provide a minimum "pension" to the current and future generations. The interim constraints will be necessary because the political economy may create pressures at any time to adjust the rules if the Fund were to be seen not to be able to fulfill its liabilities in the future, even when it may be expected to do so with sufficient likelihood.

We use the same asset classes the Norway fund has used as its equity and bond benchmarks, i.e., the same risk profiles. We add, however, some assets to allow for more strategic ALM. Importantly, we add a "negative oil" asset. For this we use the oil swap price to model the oil derivative price behavior. We also include some other assets. This broadening of investment classes allows a better hedging of the various risks Norway faces and is consistent with the Fund already allowing a broader class of assets (in 2007 there was an increase in the equity allocation limit to 60 per cent and inclusion of small-cap sector in benchmark portfolio; and recently investment in real estate was allowed). We estimate the correlations among the rates of return for the various asset classes and the correlations with oil prices, exchange and interest rates in a consistent fashion.

It is critical to have very good risk control when managing an endowment aggressively, such as done by the Yale and Harvard university endowment funds. This also applies to the Norway Fund, which has a large equity component and has had some years recently in which its equity net return was quite low (albeit in line with the overall benchmark rates of return). As noted, variance, semi-variance, and VaR approaches are inadequate to capture the type of extreme events that such funds are exposed to, as also shown recently by the failure of risk management approaches in many funds and banks. Using a downside penalty in the form of a convex nonlinear or piecewise linear function will work. We will use CVaR constraints at multiple points in time since they are very easy to understand and require no special estimation of parameters. It is also used reinsurance models to manage the tails of catastrophic loss distributions (Kreuser and Lane, 2006).

3.5 Results[23]

We will consider and analyze four possible portfolios: the current benchmark asset allocation as given by NBIM, 2008 (called "Fixed Benchmark as per NBIM"), the optimal asset allocation only considering the assets in the benchmark (called "Allowing Rebalancing"), the optimal allocation when oil funds flows are added (called "With only Oil Funds"), and the optimal allocation when a more aggressive policy is allowed (along the lines of NBIM's planned inclusion of real estate and private equity, Kjaer, 2007) , but further adding the oil swap and funds inflows from oil exports (called "With Oil Funds, Swaps, Assets'). An estimation to the current benchmark asset allocation from NBIM, 2008 is replicated in Table 1. The starting portfolio in all cases is that benchmark. We first discuss in detail the runs "Allowing Rebalancing" and "With Oil Funds, Swaps, and Assets". We then compare these two runs with two others in the section 4.1 Norway Lessons.

The first run gives the optimal allocation assuming only the benchmark asset classes, no oil swap, no alternative asset classes, and no funds flows into the fund. In the following runs, we replicate the kinds of tables that appear in NBIM, 2008 in terms of their content and currency.

[23] Besides more detail, the web-paper addresses three additional questions. The first is what else could have been done using our framework (on September 5, 2008) to improve our results and what kinds of probabilistic stress test could have been done. The second is how would our hedged portfolio have performed over the short term compared to the benchmark. Lastly, how the hedged portfolio performs with respect to the benchmark over the long term.

We note from the (scaled) third and fourth moments in Table 3 that the distribution is right-skewed so that the upside is much greater than the downside. This is due to the CVaR constraints, the details on these we give in the Table 4.

Table 3: Solution Statistics for Wealth (Billions of NOK) for "Allow Rebalancing" Case

Item	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Expected Wealth	1,933.97	1,912.84	2,253.92	3,163.82	5,582.79
Annualized Volatility %		7.45	11.61	12.56	13.30
3rd Moment		31.73	34.77	25.40	11.65
4th Moment		987.70	740.13	536.29	176.65
Annualized Krone Return		-1.09	8.50	8.74	5.56
Annualized US$ Return		4.19	5.54	5.13	4.94
Annualized NKU Return		-0.11	7.69	7.83	5.54

Table 4: CVaR Risk Analysis (Billions of NKU) for "Allow Rebalancing" Case for confidence level = 95% and expected tail loss = 20%

Item	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Expected Value of Wealth (NKU billions)	261.32	261.02	303.02	412.14	709.36
CVaR value or expected tail loss limit		208.32	242.41	329.71	567.49
Corresponding VaR Value		259.74	292.01	340.07	572.57
Expected Loss Ratio to CVaR[24]		0.49	3.63	17.49	19.28
Worst Level of Wealth		254.73	103.76	294.29	374.38
Worst Loss as a % of Expected Wealth		0.49	3.63	17.49	19.28

[24] This is the ratio of the CVaR value to the expected wealth. As it approaches 20%, the value we chose, then more events or state constraints are binding. As we go out in time, more states are bound by that constraint as the value will approach 20% but never exceed it.

The NKU returns are usually better than the US$ returns because of the currency movements. The NKU consists of 69% Euro so we can essentially think of it as Euros. The CVaR constraint becomes more binding over time on various events or states as indicated by the expected loss ratio. This is to be expected. The worst level of wealth is probably unrealistic given what is happening in reality but the asset class movements are nevertheless realistic.

In the following Table 5 we give the asset class composition and note that the model initially sells most all equity classes and puts this mostly into cash in Euros giving the portfolio a 20% weight in cash. More equity would have been sold had it not been for the constraint that no more than 25% of the portfolio can be rolled over.

Table 5: Expected Asset Class Composition in Solution (in %) for "Allow Rebalancing" Case

Asset Class	Initial prior to rebalance	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Equity – FTSE AW Europe (Euro) Equity	14.00	3.86	10.68	12.84	7.97	7.15
Equity – FTSE AW Americas (USD)	14.00	0.00	1.98	1.19	2.46	2.90
Equity – FTSE AW Asia Pacific (Yen)	6.00	6.05	10.52	7.80	9.12	8.66
Lehman Euro-Agg. 1-3 Year Bonds	15.75	15.83	1.48	0.62	0.41	0.30
Lehman Euro-Agg. 10+ Year Bonds	5.25	4.40	13.51	11.52	8.24	6.56
Lehman Global: US Treasury 1-3 Year	15.75	15.68	13.56	5.67	3.53	3.75
Lehman US Agg. A+ 5+ Year Bonds	5.25	5.23	17.46	20.82	19.26	18.97
Lehman Global: Japan 1-3 Year Bonds	6.75	6.80	.00	.01	.00	.00
Lehman Global: Japan Long Bonds	2.25	2.27	8.09	3.22	1.21	1.07
Lehman Global: UK 1-3 Year Bonds	11.25	11.22	6.02	15.10	17.37	15.44
Lehman Global: UK Long Bonds	3.75	3.74	10.49	15.12	11.34	10.91
Lehman Global: Inflation Linked Bonds	0.00	5.00	.52	.51	3.04	2.80
Euro	0.00	19.94	4.41	5.06	15.18	17.54
GBP	0.00	0.00	.01	.05	.00	1.02
USD	0.00	0.00	.05	.08	.01	1.54
Yen	0.00	0.00	1.24	.40	.88	1.41

The cash holdings show how a stochastic optimization model can do what most other models cannot do; it takes the wait-and-see position that means that it will wait to see how the uncertainty unfolds before investing in other asset classes. The other point to notice is that the model increases the allocation in long bonds over the entire horizon and reduces the equity allocation. Typical of what you would expect from a more conservative position.

In the Table 6 we summarize the portfolio composition and compare it to the NBIM benchmark. Most notably the equity proportion is much smaller than in the NBIM benchmark where 60% is used. However, in the next run when we include real estate and private equity and classify them as equity we will see the ratio approaching 60%.

Table 6: Summary Expected Portfolio Composition for "Allow Rebalancing" Case

Item	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Portfolio Composition (Billions USD)					
Overall	344.13	358.56	400.28	493.58	818.65
Equities	34.08	82.10	86.33	98.90	152.96
Fixed Income	241.44	257.51	291.73	319.98	487.34
Cash	68.61	18.95	22.22	74.69	178.35
Portfolio Composition (%)					
Overall	100.00	100.00	100.00	100.00	100.00
Equities	9.90	22.90	21.57	20.04	18.69
Fixed Income	70.16	71.82	72.88	64.83	59.53
Cash	19.94	5.29	5.55	15.13	21.79

In Table 7 we show the currency composition. Most notably the model moves out of Euro equity and short bonds and into UK long and short bonds. The model has a tendency to go for a longer bond duration as would be expected in a more conservative approach.

Table 7: Expected Currency Composition (%) for "Allow Rebalancing" Case						
Currency	NBIM Benchmark	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Euro %	35	44.02	30.07	30.04	31.80	31.55
GBP %	15	14.96	16.53	30.27	28.71	27.37
USD %	35	25.91	33.56	28.27	28.29	29.95
Yen %	15	15.12	19.84	11.43	11.21	11.13

We now expand the problem and add in real estate asset classes for Euro, USD, and Yen zones plus private equity measured in US$. We include oil swaps and funds flows based upon the oil price. We could have included other asset classes negatively correlated to the oil price and improved the results. We have assumed a deterministic declining oil stock but could have easily made that stochastic. We could also have made the extraction rate a stochastic decision variable. The solution summary is given in Table 8. The expected value we get for wealth in 2015 is 6,661 NOK instead of the 4,350 NOK expected by NBIM. But this difference can be attributed to several factors; the exact asset class set that NBIM uses, the assumptions they place on the stochastic processes, the oil funds flows they assume, and other policy constraints especially that of the risk constraints as we only use downside risk constraints and the type of model that they use. We note that the amount in oil swaps increases over time as the oil stock goes down and the price is more volatile. The third and fourth moments that the resulting distribution is skewed to the right and has fatter tails than a normal distribution. The skewness to the right is because of the CVaR constraints that only constrain the downside.

The risk analysis is given in Table 9. Here the worst loss in 2009 is more realistic and this is because more volatile asset classes come into the solution because of the funds flows due to oil. More risky asset classes can be allocated early on. The worst loss in 2009 is 49% of the portfolio. The worst loss in 2025 is around 50% of the expected value of the portfolio but the expected loss of the portfolio is at 20% of the expected portfolio value at the 95% confidence level because of the CVaR constraint. The first period returns are negative but the later returns make up for the early low returns.

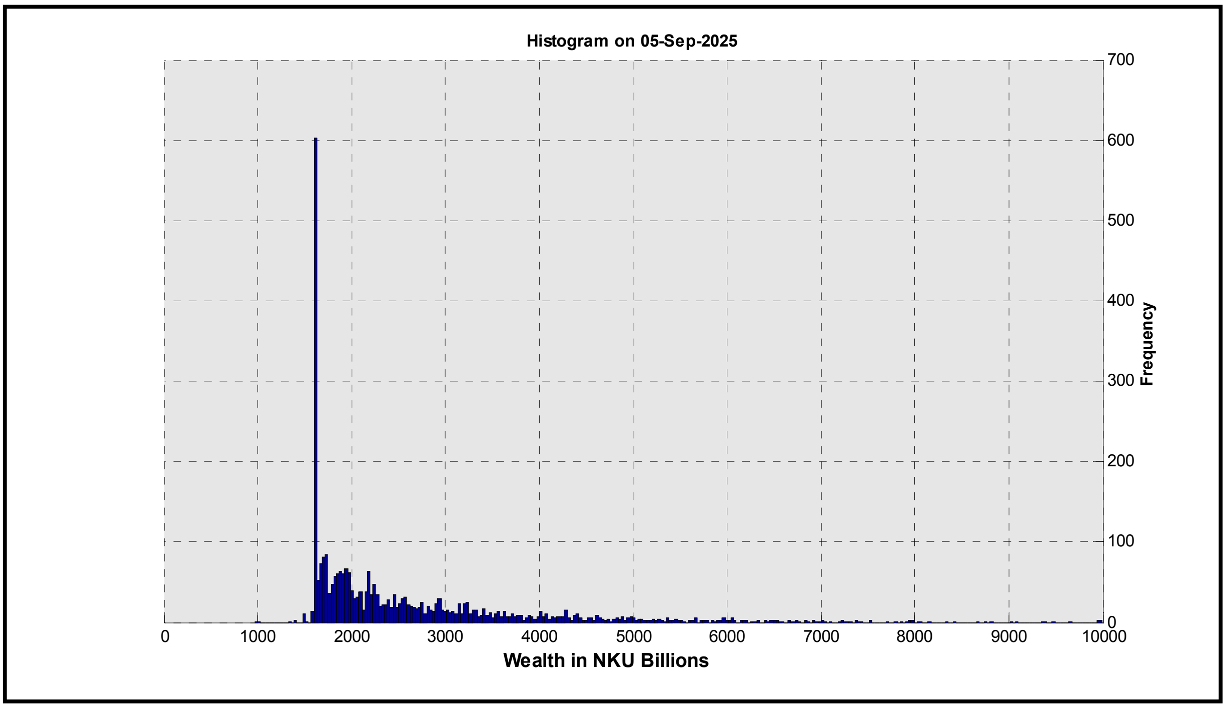
Table 8: Solution[25] for Expectations of Wealth (Billions of NOK) for "With Oil Funds, Swaps, Assets" Case					
Item	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Expected Wealth	1,934.01	2,201.22	3,404.62	6,661.86	15,382.24
Funds Flows over the Period	0.00	349.97	716.30	1,231.08	1,664.75
Swaps Flows over the Period	0.00	0.00	12.83	30.85	100.39
Annualized Volatility %		8.94	8.24	7.42	8.53
3rd Moment		3.28	0.53	2.34	3.17
4th Moment		32.39	19.72	10.70	8.90
Annualized Krone Return		-2.11	9.90	10.48	6.31
Annualized USD Return		3.68	6.80	6.64	5.49
Annualized NKU Return		-0.97	9.12	9.47	6.07
Sharpe Ratio in NKU		-0.13	1.59	1.32	.72

Table 9: CVaR Risk Analysis for "With Oil Funds, Swaps, Assets" Case in NKU Billions For confidence level = 95% and expected tail loss = 20%					
Item	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Expected Wealth (NKU billions)	261.33	300.89	458.44	869.49	1,962.17
CVaR value or expected tail loss limit		240.71	366.75	695.60	1,569.74
Corresponding VaR Value		281.80	431.06	713.17	1,619.20
Expected Loss Ratio to CVaR		6.34	5.97	17.98	17.48
Worst Level of Wealth		153.44	260.62	577.61	968.07
Worst Loss as a % of Expected Wealth		49.00	43.15	33.57	50.66

[25] We note that returns are computed prior to funds flows from oil and therefore based only upon portfolio returns but CVaR constraints are computed after funds flows.

The histogram in Figure 4 shows how well the downside CVaR constraints have reduced the worst outcomes, i.e., the left side of the distribution.

Figure 4: Histogram of wealth in NKU on 5-Sep-2025



The Table 10 gives the asset class composition in the solution. In this case, the model initially sells Euro and Yen short bonds and some US equity and invests in Yen and Euro real estate asset classes. More equity would have been sold off if it had not been for the 25% rollover constraint. The Yen area real estate is positively correlated with equities and in the short-term is decreasing in value. But the long-term growth is good and the appreciation with respect to the NKU is also good. The model detail shows that the Yen real estate asset class purchased in 2008 is held in many scenarios until 2025. Private equity is not purchased until 2009 but from then on a substantial allocation goes to that asset class. The allocations may be higher than desirable and some upper limit should be put on these asset classes as a policy constraint. Again in terms of duration, the model favors long-term bonds over short-term bonds.

This strategy is consistent with a more aggressive allocation facilitated by funds flows from oil. The Table 11 shows that equity approaches 60% as we add in real estate and private equity.

Table 10: Expected Asset Class Composition in Solution (in %) for "With Oil Funds, Swaps, Assets" Case

Asset Class	Initial prior to rebalance	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Equity – FTSE AW Europe (Euro) Equity	14.00	14.07	9.99	1.77	1.98	1.52
Equity – FTSE AW Americas (USD)	14.00	11.02	.01	2.03	.85	.88
Equity – FTSE AW Asia Pacific (Yen)	6.00	6.05	7.36	3.51	5.55	4.88
Lehman Euro-Agg. 1-3 Year Bonds	15.75	.86	1.63	.06	.01	.01
Lehman Euro-Agg. 10+ Year Bonds	5.25	5.27	11.69	14.96	8.10	4.83
Lehman Global: US Treasury 1-3 Year	15.75	15.68	7.99	.57	1.01	.85
Lehman US Agg. A+ 5+ Year Bonds	5.25	5.23	11.68	15.32	17.91	15.06
Lehman Global: Japan 1-3 Year Bonds	6.75	0.00	.00	.01	0.00	.01
Lehman Global: Japan Long Bonds	2.25	1.92	1.57	.11	1.42	.94
Lehman Global: UK 1-3 Year Bonds	11.25	11.22	.02	.14	.66	.46
Lehman Global: UK Long Bonds	3.75	3.74	10.56	9.74	13.98	10.60
Lehman Global: Inflation Linked Bonds	0.00	.00	3.59	3.39	4.94	4.50
DJ US Real Estate Total Return	0.00	0.00	5.97	5.46	.96	.86
DJ Euro STOXX Real Estate Total Return	0.00	9.79	7.81	10.81	9.21	7.38
DJ STOXX Asia Pacific Total Return	0.00	15.15	8.82	10.36	4.22	3.39
Private Equity Total Return	0.00	0.00	10.52	19.09	19.07	18.44
Euro Cash	0.00	0.00	.70	2.47	10.71	17.81
GBP Cash	0.00	0.00	.05	.05	.00	.83
USD Cash	0.00	0.00	.00	.03	.00	6.18
Yen Cash	0.00	0.00	.05	.13	.14	.60

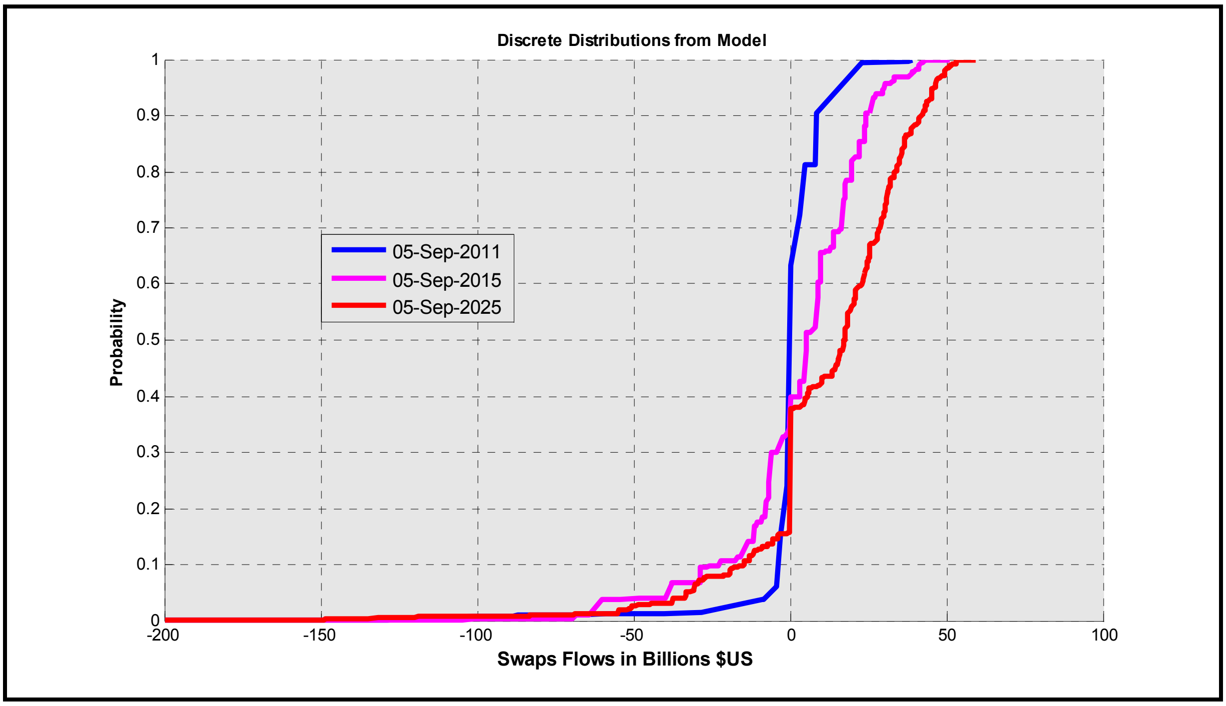
Table 11: Optimal Portfolio Composition for "With Oil Funds, Swaps, Assets" Case					
Item	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Portfolio Composition (Billions USD)					
Overall	344.14	414.85	605.12	1,033.81	2,212.40
Equities	192.98	206.35	321.61	428.59	823.06
Fixed Income	151.16	204.50	266.78	494.80	820.39
Cash	0.00	4.01	16.73	110.43	568.95
Portfolio Composition (%)					
Overall	100.00	100.00	100.00	100.00	100.00
Equities	56.08	49.74	53.15	41.46	37.20
Fixed Income	43.92	49.29	44.09	47.86	37.08
Cash	0.00	0.97	2.76	10.68	25.72

The expected currency composition is given in the Table 12. The USD percentages are high but this has more to do with the fact that the private equity is measured in USD.

Table 12: Expected Currency Composition (%) for "With Oil Funds, Swaps, Assets" Case						
Currency	NBIM Benchmark	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Euro %	35.00	30.00	31.82	30.07	30.02	31.55
GBP %	15.00	14.95	10.63	9.93	14.64	11.88
USD %	35.00	31.93	39.75	45.89	44.02	46.76
Yen %	15.00	23.12	17.80	14.12	11.33	9.80

Figure 5 provides information on the distribution of USD flows due to oil swaps in each period.

Figure 5: Distribution of flows due to USD swaps in each period.



In the web-paper we provide more details and more analysis. This includes details on the CVaR constraints, the sensitivity of buy and sell prices, sensitivity to policy constraints, implied prices on scenarios used to price other instruments, more details on the oil price funds flows and swaps, etc. We also investigate the dynamics. We also look at the relationship between returns on asset classes and break them up between returns on the assets and returns on foreign exchange.

4. Discussion and general applicability

4.1 Norway lessons

In this section we compare different solutions and in that way emphasize the importance of dynamics, CVaR constraints, and general stochastic optimization with inequalities. We could compare our framework with the mean-variance solution, but that is a very different approach. We therefore add what we call the "Fixed Benchmark as per NBIM". In this case we assume that the proportions of asset classes remain always fixed to the benchmark proportions and compute the expected wealth over the same scenario tree as for the first case analyzed in this paper. The next case we add is the "With Only Oil Funds" case, where the flows into the fund are based on a stochastic oil price, but the asset classes remain the same as in the "Fixed Benchmark as per NBIM" case. We can compare these cases then to the solutions presented in the previous section, the "Allowing Rebalancing" case and the "With Oil Funds, Swaps, and Assets" case. All cases are summarized in Table 13. Table 14 contains Sharpe ratios for all cases, the "Downside Sharpe Ratio" (DSR), and statistics on the oil price. The Sharpe ratio is computed using a risk free rate of zero. The DSR[26] measures the downside deviation and is adjusted by the probability of the downside. In the event that the process is normal, then the DSR and the Shape ratio are equivalent. In the event that the process is skewed or constrained to the upside, the DSR is a more useful comparator because it does not penalize for upside returns.

The "Fixed Benchmark" case should be suboptimal to the "Allowing Rebalancing" case. But this is not completely obvious as the Fixed Benchmark case does not have to satisfy the constraints that the Allowing Rebalancing case does. In spite of this, the Allowing Rebalancing case dominates the Fixed Benchmark. The Allowing Rebalancing case dominates the Fixed Benchmark case in wealth in every period in NOK and NKU and its Sharpe ratios measured in NKU also dominate in every period. We show the distributions in Figure 6 and see that the Allowing Rebalancing case stochastically dominates the Fixed Benchmark case. In fact it stochastically dominates in every rebalancing period except for September 5, 2009. The Rebalancing case buys assets initially that can be held in whole or in part until 2025 and will shift them around as the economic and financial environments deems necessary. The interim CVaR constraints keep the downside risks from becoming too great at any rebalancing period and the distribution is shifted to the right.

Comparing the "With Only Oil Funds" case and the "With Oil Funds, Swaps, and Assets" cases, there is not much difference until 2009, but then values begin to diverge. By 2011 the "With Oil Funds, Swaps, and Assets" case dominates the "With Only Oil Funds" case in all indicators and as shown in the distributions in Figure 7 by 2025. The model suggests investments in the new asset classes, even though they won't pay off until later year. At the same time, CVaR constraints prevent the downside risks from becoming too great. The addition of the asset Oil Swaps allows for better risk diversification in later years. As such, the SWF helps to reduce the impact of oil price volatility on Norway's overall economy, i.e., it acts as an insurance mechanism. As in the previous case, we see the stochastic dominance exhibited in Figure 7. And again, this stochastic dominance holds in every rebalancing period except September 5, 2009.

[26] See Ziemba znd Ziemba 2007 for a discussion of similar ratios.

Figure 6: Distribution of "Fixed Benchmark as per NBIM versus "Allowing Rebalancing" in 2025

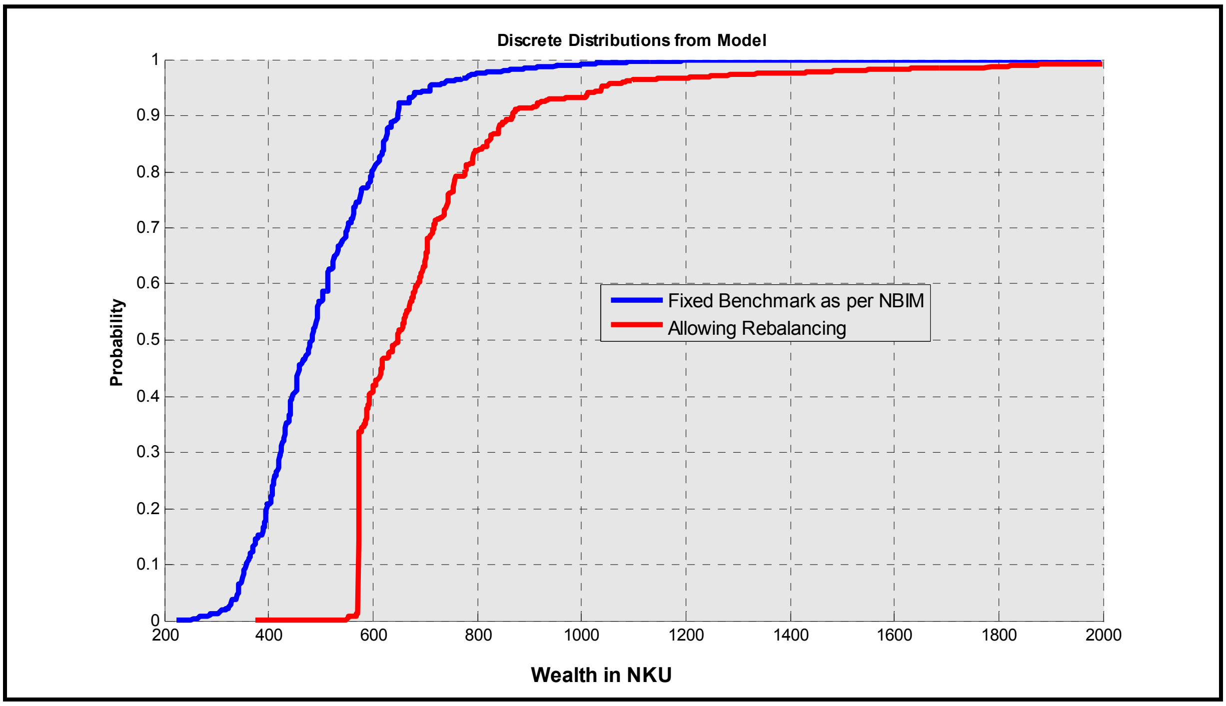


Figure 7: Distribution of "With only Oil Funds" versus "with Oil Funds, Swaps, Assets" in 2025

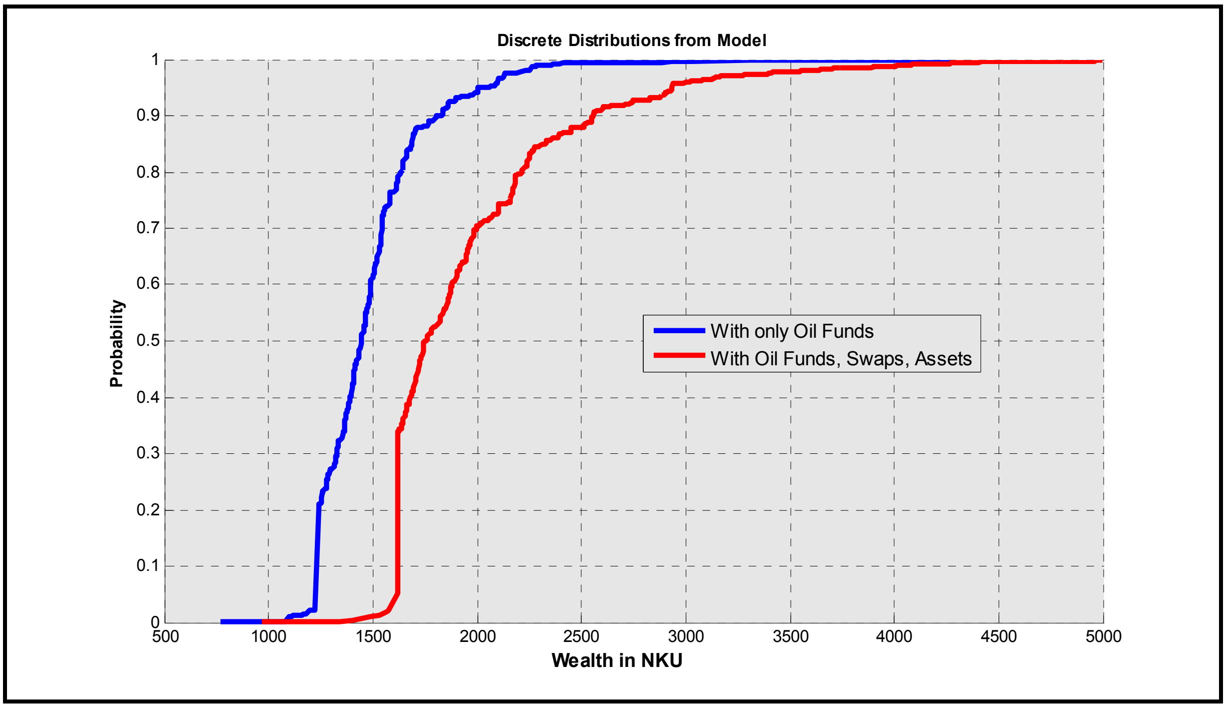


Table 13: Comparison Statistics for Four Problems					
Item	05-sep 2008	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Expected Wealth (Billions of NOK)					
Fixed Benchmark as per NBIM	1,935.59	1,892.51	2,099.87	2,682.83	3,943.38
Allowing Rebalancing	1,933.97	1,912.84	2,253.92	3,163.82	5,582.79
With only Oil Funds	1,933.72	2,231.22	3,287.91	5,849.82	11,679.88
With Oil Funds, Swaps, Assets	1,934.01	2,201.22	3,404.61	6,661.86	15,382.24
Expected Wealth (Billions of NKU)					
Fixed Benchmark as per NBIM	261.54	258.31	281.87	349.26	501.22
Allowing Rebalancing	261.32	261.02	303.02	412.14	709.36
With only Oil Funds	261.29	304.82	443.22	763.07	1,488.66
With Oil Funds, Swaps, Assets	261.33	300.89	458.44	869.49	1,962.17
Annualized Rate of Return in % (based on NKU)					
Fixed Benchmark as per NBIM		-1.23	4.41	5.43	3.53
Allowing Rebalancing		-.11	7.70	7.83	5.36
With only Oil Funds		0.34	6.67	6.76	4.38
With Oil Funds, Swaps, Assets		-0.97	9.13	9.48	6.07

4.2 Applying the framework to other types of SWFs

There are many types of sovereign wealth funds and this framework can be applied to these cases with some adaptations. For example, national pension funds would want to invest safely and maximize long-term wealth, yet have sufficient liquidity to make interim payments on liabilities. In this case we would want to add equations (15) and (25) for target at risk, or often called funding ratio in the case of pension funds. This would be applied at each rebalancing point in time in the model as CVaR constraints. If the fund has explicit expectations of the its liabilities, we could use preference functions (4) and (5) to obtain a solution with a density function as close as possible to the value of expected liabilities. This would produce a solution where the pension fund can meet its obligations safely. We could also apply a CVaR constraint to the shortfall in wealth, especially if some risky asset classes were included.

Table 14: Statistics for Portfolio Ratios and Oil				
Item	05-Sep 2009	05-Sep 2011	05-Sep 2015	05-Sep 2025
Sharpe Ratio in NKU				
Fixed Benchmark as per NBIM	-0.21	0.57	0.64	0.42
Allowing Rebalancing	-0.03	1.41	0.69	0.44
With only Oil Funds	0.06	0.91	1.11	0.75
With Oil Funds, Swaps, Assets	-0.13	1.59	1.32	0.72
DSR in NKU				
Fixed Benchmark as per NBIM	-0.20	0.61	0.76	0.55
Allowing Rebalancing	-0.05	1.48	1.51	1.08
With only Oil Funds	0.07	0.91	1.44	1.01
With Oil Funds, Swaps, Assets	-0.14	1.64	1.79	1.30
Oil Price in NKU for Oil Stock				
Annualized Expected Return	18.32	5.28	0.39	-0.49
Volatility %	26.81	20.61	16.74	15.35
Sharpe Ratio	.68	.26	.02	-.03

Commodity stabilization funds would want to manage their resources more like central bank reserves do for safety, liquidity, returns, and stability. To incorporate safety and returns, we could adapt equations (4) and (5) for the objective function, to obtain more narrow distributions for targeted wealth levels, while incorporating the effects of stochastic commodity prices (like we did for oil prices). Liquidity constraints, equations (17) and (26) plus associated CVaR constraints could be made a function of the stochastic commodity prices to ensure funds are available when prices fall. Stability constraints such as (18) and (27) could be added to smooth the degree of asset sales over time.

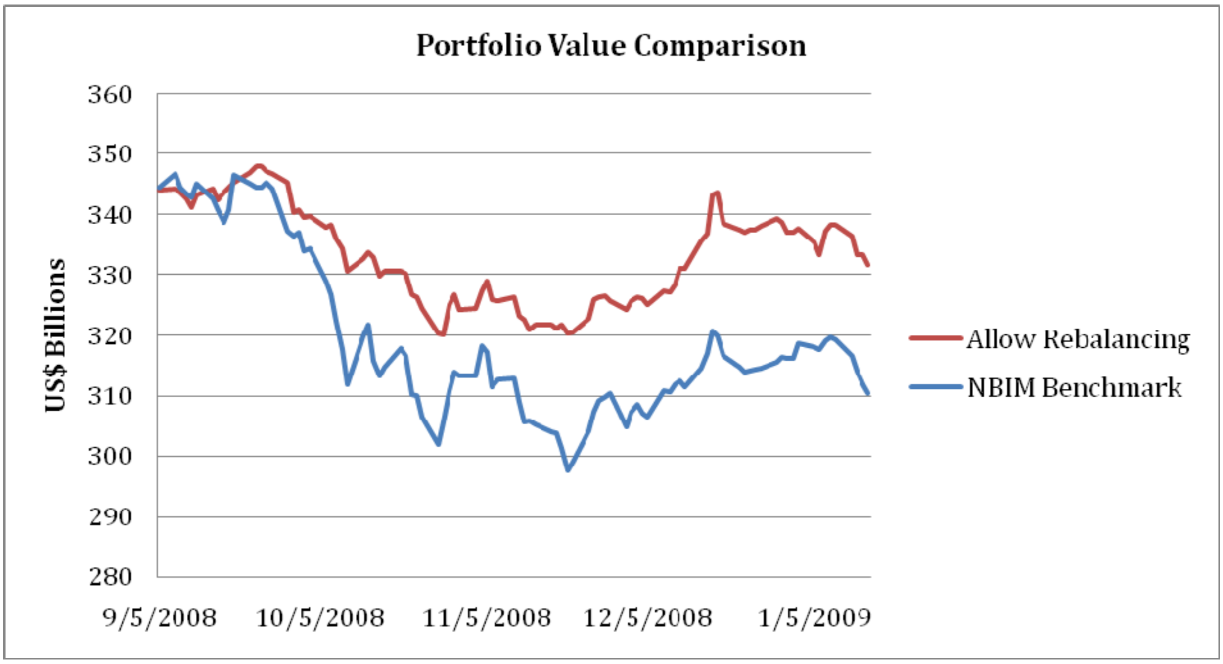
Development funds could be treated as endowment funds, as in this paper, with the addition of some known or estimated liabilities, similarly to national pension funds. In addition, because of legal or other policy parameters, any of the following could be added as policy constraints: minimum level of exposure to a currency, maximum level of an asset class as a percent of wealth, maximum percentage of portfolio rollover, and unrealized losses at risk, all defined in this paper.

4.3 Epilogue

The models were run using only historical data before September 5, 2008. No expert views were imposed on the model. We updated the data to January 15, 2009 to see how the model recommendations would have performed. The model was constrained from rolling over more than 25% of the portfolio initially. The graphs of the portfolio values from September 5 to January 15 are given in Figure 8. If 25% of the portfolio could have been rolled over along with the recommendations of the model Allowing Rebalancing, US$ 20 billion would have been saved by January 15.

This model is a strategic model and normally it is not meant to control short-term risk. Generally these models prevent extremes from occurring. This along with the fact that the model allows for regime changes and considers stochastic processes in the short-term and in the long-term means that the model still moves out of equities because equities were still declining in the short-term. The case for the model with oil funds and swaps is more complicated and it didn't perform as well in the short-term. The reasons for that are analyzed in the web paper.

Figure 8: Comparison of Portfolio Values



5. Conclusions

We have presented a framework for sovereign wealth fund investment and risk management and have shown that it is readily implementable and applicable. We have calculated a benchmark based upon our framework and have compared it to that used by the Government Pension Fund – Global of Norges Bank based upon their quarterly report NBIM, 2008. We have included similar tables and analysis in our paper as is contained in that report. We have extended our analysis further and examined the case when additional asset classes such as property, private equity, and oil swaps are added and when oil funds flows are added along with stochastic oil prices. The analysis was done using data as of September 5, 2008 using only historical information and no expert views. When we estimated the portfolio value of the benchmark recommended by the model to January 15, 2009, US$ 20 billion would have been saved over the NBIM benchmark.

Our framework combines tactical asset allocation considerations with broad macroeconomic, macro-prudential risk, and sovereign investment management considerations. Our investment and risk management framework allows for very general objective definitions and does not restrict the class of eligible stochastic processes or limit the investment universe. We showed that many assets and liabilities can be included and how to mitigate the curse-of-dimensionality by generating good scenario trees over several periods. We have included the oil stock and stochastic prices in the model and indicated how we could include the optimal extraction rate and stochastic transfers to the government budget.

We see our approach as an important complement to the various ALM-tools currently being used by sovereign wealth funds. Although our approach is more demanding than other approaches in terms of analytical modeling regarding objectives, constraints and assumptions, we think this is worthwhile for several reasons. First of all, strategic allocation and the formation of benchmarks can be one of the most important determinants of portfolio returns and overall investment management objectives. It is therefore important that the strategic analysis be done incorporating all the risks and other factors in an integrated fashion. Second, a dynamic framework that allows for rebalancing is very important for benchmark creation especially in the presence of transaction costs, constraints over time, changing financial and economic regimes, and changing cash flows. It allows for incorporating longer-term effects such as mean-reversion of exchange rates, changing correlations, extreme events and contingent liabilities, and the effects of instruments such as longer duration assets and derivatives. Third, in the case of sovereign wealth funds especially, it is crucial that the formulation of objectives and risk constraints captures the various considerations regarding the multiple risks at all points in time. Our framework allows multiple risk measures to be put on multiple factors at different times and to only constraint the downside risks using CVaR constraints. Lastly, the existing approaches used often do not provide the richness of sensitivity and stochastic stress tests provided by our framework that are needed to address the various considerations facing SWF managers.

Our approach has also important advantages because it requires a very explicit process for model development that will help clarify the strategic aspects involved in risk management. Often, these strategic aspects, including linkages between macro-, micro-economic and financial risks are left out or treated only implicitly in other approaches. At the same time, the use of numerical approaches to solving using a dynamic stochastic optimization framework allows for analytical rigor while maintaining ample modeling flexibility. As implemented, by using CVaR constraints and objectives that shape densities as major tools, the model retains many intuitive features of risk attractive to policy makers and doesn't require estimating utility functions. The approach not only provides the complete density functions of outcomes of a selected portfolio (instead of just summary measures), but also defines criteria for the shape of density functions in terms senior management can understand. Then, satisfying those criteria, our approach determines the strategic portfolio allocation that is optimal over time with respect to an objective—such as returns or the likelihood of meeting a target, such as a funding ratio or other liabilities. This intuitive approach can be supported at each level in an institution, while its flexibility allows adaptation to the unique requirements of each individual fund[27].

Annex

Model Variables	**Definition**
ALPHA	Alpha variable in definition of CVAR constraints. It will take on the appropriate VaR value in an optimal solution.
$A_{i,c}^{t,e,\tau,\varepsilon}$	Holdings at time τ and event ε of asset *i* in currency *c* acquired at time *t* and event *e*.
$CASH_c^{t,e}$	Cash in currency *c* at time *t* and event *e*..
PRF	Value of preference function.

[27] "Stochastic programming is the ideal tool for synthesizing the firm-wide risk analysis into firm-wide risk management. Finally post-optimality analysis of the models will allow us to develop of a firm-wide performance measurement system and allowing for the decomposition of risk measures and performance measures into components attributable to the various underlying risk factors" in Kouwenberg and Zenios, "Stochastic Programming Models for Asset Liability Management", page 297 of Zenios and Ziemba, 2007, Vol.1.

$SG_1^{t,e}$, $SG_2^{t,e}$, $SG_3^{t,e}$	Variables for segmenting the piece-wise linear-quadratic form.
$SHF^{t,e}$	Shortfall variable representing the shortfall below some target
$SWAPS^{t,e}$	Oil swaps in time *t* and event *e*.
$TW^{t,e}$	Total wealth in time *t* and event *e*.
$TRS^{t,e}$	Total transaction costs.
$V_{c,d}^{t,e}$	Transfer of currency from *c* to *d* at time *t* and event *e*.
W^t	Total expected wealth at time t in the numeraire currency.

Tree Variables	**Definition**
δ^t	Discount factor at time *t*. Used in the preference function.
$\gamma_c^{t,e}$	Exchange rate in terms of the numeraire at time *t* and event *e*.
$\eta_{i,c}^{\tau,t,e}$	The price adjustment for asset *i* in currency *c* acquired at time τ and marked-to-market at time *t* and event *e*.
$\pi^{t,e}$	The probability that event *e* will occur in time *t*.
$op^{t,e}$	Oil price at time *t* and event *e*.
$std^{t,e}$	Short-term debt at time *t* and event *e*.

Parameters/constants	**Definition**
$cliq^t$	Liquidity portfolio requirements in time *t*.
$ina_{i,c}$	The beginning portfolio in asset *i* and currency *c*.
$liqb$	Initial liquidity portfolio.
ma_c^t	Minimum activity policy constraint for currency *c* and time *t*.

$mat_{i,c}$	Maturity of asset *i* in currency *c*.
$nret^t$	Minimum return (realized or unrealized) on the portfolio for time *t*.
$\left[p^t, q^t\right]$	Interval for each time period *t* in which one wants to push the probability mass.
r_1^t, r_2^t	The beginning and ending slopes of the linear parts of the piecewise linear-quadratic at time *t*. The values are set to push more or less probability mass into the interval $\left[p^t, q^t\right]$ for each time *t*. Typically we use $\left[r_1, r_2\right] = [100,.01]$ as a starting interval for maximization.
rc	Return on cash.
$stock^t$	Oil stock in time interval *t*.
tc_c	Transaction cost for currency transfers in currency *c*.
tca_i	Transaction cost of purchase of asset *i*.
tcs_i	Transaction cost of sale of asset *i*.

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