Super-Exponential RE Bubble Model with Efficient Crashes

Jerome Kreuser[1] and Didier Sornette[2]

ETH Zurich

October 2017

(Submitted 22 Oct 2017) (Revision 4 June 2018)

Abstract

We propose a dynamic Rational Expectations (RE) bubble model of prices with the intention to exploit it for and evaluate it on optimal investment strategies. Our bubble model is defined as a geometric random walk combined with separate crash (and rally) discrete jump distributions associated with positive (and negative) bubbles. We assume that crashes tend to efficiently bring back excess bubble prices close to a "normal" or fundamental value ("efficient crashes"). Then, the RE condition implies that the excess risk premium of the risky asset exposed to crashes is an increasing function of the amplitude of the expected crash, which itself grows with the bubble mispricing: hence, the larger the bubble price, the larger its subsequent growth rate. This positive feedback of price on return is the archetype of super-exponential price dynamics, which has been previously proposed as a general definition of bubbles. Our bubble model also allows for a sequence of small jumps or long-term corrections. We use the RE condition to estimate the real-time crash probability dynamically through an accelerating probability function depending on the increasing expected return. The facts that our jump process is related to the price process thought the normal price and the probability distribution can change over time make it different from existing bubble models. After showing how to estimate the model parameters, we examine the optimal investment problem in the context of the bubble model by obtaining an analytic expression for maximizing the expected log of wealth (Kelly criterion) for the risky asset and a risk-free asset. We also obtain a closed-form approximation for the optimal investment. We demonstrate, on seven historical crashes, the promising outperformance of the method compared to a 60/40 portfolio, the classic Kelly allocation, and the risky asset, and how it mitigates jumps, both positive and negative.

Keywords: financial bubbles, efficient crashes, positive feedback, rational expectation, Kelly criterion, optimal investment

JEL: C53, G01, G17

[1] RisKontroller Global LLC and Senior Researcher ETH Zürich – Chair of Entrepreneurial Risks.
[2] ETH Zürich, Dept. of Management, Technology and Economics, Zürich, Switzerland, also at the Swiss Finance Institute, c/o University of Geneva, Switzerland.

1 Introduction

The last financial crisis revealed serious flaws in economic modelling and in the use of mathematical and engineering models in finance, in particular with respect to the occurrence of bubbles, crashes and crises. The present article contributes to enriching the understanding of financial markets by proposing a simple bubble and crash model, which can be calibrated and made operational in portfolio investments. The model stresses the importance of positive feedbacks, the tendency for financial markets to self-correct only at long time scales (years to decades) while exhibiting significant departure from "normality" at short times (day, months and even years).

In academia, discussion on financial bubbles often start with a reference to the Efficient Market Hypothesis (EMH), which in essence states that prices of financial assets properly reflect underlying economic fundamentals. Financial bubbles and the crashes that frequently follow them are arguably the most vivid challenge to the EMH. Here, we define a bubble as a period of unsustainable growth when the price of an asset increases ever more quickly in a way not justified by fundamental valuation. A strand of literature has thus developed to detect deviations from the elusive fundamental value, with an extensive econometric literature on the identification of bubbles, see e.g. (Homm and Breitung, 2012; Phillips et al., 2015; Vogel and Werner, 2015). Another branch of the literature has been concerned with the possible generating mechanisms, in particular addressing the paradoxes posed by the apparent arbitrage opportunities provided by persistent overpricing during bubble regimes, see e.g. the reviews (Kaizoji and Sornette, 2010; Brunnermeier and Oehmke, 2013; Xiong, 2013). The present article focuses on the second part concerned with the development of a suitable theoretical framework to model financial bubbles, which can be exploited to develop crash- and rally-aware optimal portfolios.

The present article elaborates on a number of works developed by our group, which start with an analysis of the price behavior in times of a possible bubble. As in Evans' model, one assumes that there is the possibility of a crash in a bubbly market. The assumption that the bubble is a rational expectation bubble gives a relationship between the price process and a possible crash. Some of the main concepts that are needed to understand the behavior of financial markets are social imitation, herding, self-organized cooperativity and positive feedbacks, which leads to super-exponential, unsustainable growth of the price process (Johansen et al., 1999; 2000; Sornette, 2003; 2014; Johansen and Sornette, 2010; Jiang et al., 2010; Sornette and Cauwels, 2015]. We note that super exponential growth during a bubble has been confirmed in a model-independent analysis of real stock market data (Leiss et al., 2015) as well as in price formation experiments (Hüsler et al., 2013).

With this background, our bubble model has the following important properties:

1. It is a Rational Expectations model.
2. Prices temporarily deviate from a fundamental value or "normal price" process.
3. It is mildly explosive when the crash/rally probabilities are taken as average.
4. It can become super-exponential, following a path that would end with finite time singularities when probabilities are computed dynamically in a positive or negative bubble. The presence of crashes prevents actually reaching the finite-time singularities.
5. It never stops even on negative bubbles.
6. The price stochastically oscillates around a normal price until it randomly begins to grow or decline and then accelerate to a bubble (positive or negative).

It also has the following secondary properties:

1. The price growth converges in the limit to that of the normal price process.
2. As a consequence of the crashes and rallies together with the transient super-exponential phases, the price oscillates between positive and negative bubbles.
3. There is no upper or lower bound on the log of the price.
4. It combines a geometric random walk with a discrete Poisson distributions of crashes/rallies.
5. The crash/rally distribution sizes allow for over- and under-shooting the normal price.
6. Prices never become infinite as the crash probability becomes one before that happens.
7. It shows how bubbles can be spontaneously initiated and terminated.
8. It can be tested empirically by implementing an optimal investment method, which demonstrates a superior bubble mitigation performance.
9. It is arbitrage free.

In this model, a bubble begins because a random fluctuation has a large enough deviation from a normal price to throw it into bubble state whereby it may continue to accelerate because, in the presence of positive feedback, it takes larger correcting random fluctuations to bring the price sufficiently back down. This is conceptually similar to the mechanism put forward by Harras and Sornette (2011) in which bubbles originate from a random lucky streak of positive news that, due to a feedback mechanism of these news on the agents' strategies, develop into a transient collective herding regime. The bubble continues until it probabilistically bursts, moving back toward the normal price level according to a distribution of crash sizes.

Our bubble model suggests that investment in the bubble is rational given the expectation that players can sell off at a higher price in the future before the bubble bursts. Yet, some players may get out as the probability increases beyond their risk threshold resulting in a plateau of prices before bursting. The phenomena of acceleration and plateau are those that we capture in our bubble model.

Our paper is organized as follows. We begin with section 2 by defining the rational expectations model with efficient crashes. Section 3 shows how to calculate parameters of the model, particularly jumps and the separation of the geometric random walk from jumps and the jump probability. Section 4 presents the calculation of the asset allocation of the optimal investment problem based on Kelly's criterion or maximum expected log of wealth assuming average jumps and jump probability. Section 5 extends the bubble model to when the crash probability is determined dynamically through the rational expectations condition on real data. This generates a super-exponential bubble model with underlying finite time singularities (which however are never attained due to the agency of crashes and rallies). We show how to estimate the probability of a crash/rally. Section 6 gives seven examples on real financial time series that exhibited bubbles and crashes. We show how our bubble model coupled with its specific Kelly portfolio allocation allows us to mitigate the crashes and provides superior performance. Lastly, Section 7 concludes and discuss further work.

2 Rational Expectation bubble model with "efficient crashes"

2.1 Preliminary considerations and conditions

This paper is concerned with a price process $p_t > 0$ and an associated bubble process B_t that satisfies three conditions:

Condition 1: **Rational Expectations.** We say a price process $p_t > 0$ satisfies the rational expectations condition if $E\left(\ln\left(\frac{p_{t+1}}{p_t}\right)\right) = r_D \quad \forall t$. We call r_D the discount rate[3]. We assume it is constant, but this will be relaxed later.

Condition 2: **Efficient Crashes.** We say that a price process $p_t > 0$ with $p_0 = 1$ (for simplicity) satisfies the efficient crash condition if it has a bubble process $B_t > 0$ and if there exists a "normal price" process $N_t = p_0 \exp(r_N t)$ such that $p_t = N_t B_t$ with $\lim_{t\to\infty} \frac{1}{t} E\left[\ln\left(B_t\right)\right] = 0$.

Condition 3: **Investment/Economic Rationalization.** The investment problem is concerned with obtaining an optimal asset allocation. The economic problem or financial economics problem is concerned with models of the market, explaining how bubbles grow and collapse, and the impact of bubbles on asset prices. This condition allows us to apply processes satisfying Conditions 1 and 2 to both problems. When we assume that the rates r_D and r_N vary over time, we call them $r_{D,t}$ and $r_{N,t}$. They can be random but must satisfy the condition that $E\left(r_{D,t}\right) < \infty$ and $E\left(r_{N,t}\right) < \infty$.

The process N_t is equivalent to the "normal price" process of Merton (1971). These conditions imply that $\lim_{t\to\infty} \frac{1}{t} E\left[\ln\left(p_t\right)\right] = r_N = r_D$ or that the average return on the price converges to r_N . The pair $\left(N_t, B_t\right)$ need not be unique. B_t may consist of crashes, rallies, jumps, or slow corrections back to the normal price. By construction, the efficient crashes condition relates the crash amplitude $\ln\left(B_t\right) = \ln\left(\frac{p_t}{N_t}\right)$ to the mispricing difference between the bubble price and the normal price.

The efficient crash condition does not imply that a crash moves the price exactly back to the normal price but rather that in general the price will converge toward the normal price and oscillate about it via the cumulative actions of the crashes in the presence of the burgeoning bubble process.

---

[3] This is not to be confused with the interest rate charged to commercial banks by the Federal Reserve, although it could refer to that.

Proposition 1. If a price process $p_t > 0$ satisfies the rational expectations condition and the efficient crash condition, then the bubble process B_t fluctuates about the normal price N, never terminates, does not need to correct exactly back to N, is finite, and accelerates in proportion to the distance from the normal price N_t.

Proof:

We have $\ln\left(\frac{p_{t+1}}{p_t}\right) = r_N + \ln\left(\frac{B_{t+1}}{B_t}\right)$ and since the price process satisfies the rational expectation condition, $E\left[\ln\left(\frac{B_{t+1}}{B_t}\right)\right] = r_D - r_N$. By definition $\ln\left(\frac{B_{t+1}}{B_t}\right) = \ln\left(\frac{p_{t+1}}{N_{t+1}}\right) - \ln\left(\frac{p_t}{N_t}\right)$. In other words, the bubble process accelerates, whether a positive or a negative bubble, proportional to the asset mispricing. And it crashes back towards or eventually corrects to the normal price.

QED.

The study of bubbles (rational expectations or not) has tended to focus on two aspects; the investment problem and the financial economic implications, see Davis and Lleo (2013a). We combine both aspects in our bubble model. It is in the context of evaluating our bubble model performance in optimal investment in mitigating crashes and taking advantage of rallies and explaining how bubbles begin and end.

Bubble models have classically considered prices and dividends. Then a bubble is defined as when an asset's price exceeds the discounted value of future expected cash flows, which can be prices plus dividends.

However, in our bubble model, we assume total returns and similarly in historical price time series. In our bubble model, when the average normal price rate converges to the discount rate, our current price is always the discounted value of the expected future price and the average expected return on the bubble component converges to zero. Therefore, we do not have the usual difficulties in rational expectations bubble models requiring the bubble component being exactly equal to the asset's required rate of return or issues in an upper bound on the price. See for example, Scherbina, 2013.

Usually a price process with a bubble is written $p_t = N_t + B_t$ as the sum of a fundamental and a bubble component. If the fundamental price process $N_t = p_0 \exp(r_N t)$ has $B_t > -N_t$ with $\left|\ln\left(1+\frac{B_t}{N_t}\right)\right| < \infty$, then it satisfies the efficient crash condition if $\lim_{t\to\infty}\frac{1}{t}E\left[\ln\left(1+\frac{B_t}{N_t}\right)\right]=0$.

In the bubble model of Blanchard and Watson (1982), when the fundamental value is added to the bubble component, the bubble component must also change at the discount rate. However, if the asset is redeemable at a finite time in the future, the future bubble component must be zero. This together with the condition that negative bubbles cannot exist make it impossible to satisfy our Condition 2. It also means that, when we apply our bubble model to bonds for example, we are applying it to a total return index for bonds, which may have an infinite life.

The paper most related to ours is (Yan, Woodward, and Sornette, 2010), which studied how to infer the fundamental value and crash nonlinearity from bubble calibration. In that paper, they posed a bubble model in the form of an efficient crash model where the bubble component takes the form $-\kappa\frac{(p_t - N_t)^\gamma}{p_t}$. Here we use a similar expression $-\kappa \ln\left(\frac{p_t}{N_t}\right)$ for the crash amplitude when it occurs at time *t*.

2.2 Model definition

We define the following set of variables:

Δt = discrete time interval $[t, t+1]$.

p_t = price of the risky asset at time *t*.

$\bar{r}_t$ = expected return of the risky asset on Δt when there is no crash or rally.

σ = standard deviation on Δt of the geometric random walk price process.

ε_t = sample from a standard normal distribution at time *t*.

r_D = discount rate of the asset price on Δt.

r_N = growth rate of the "normal price" on Δt.

r_f = risk-free rate on Δt.

p_0 = starting price of the risky asset.

$N_t = p_0 \exp(r_N t)$: this defines the normal price process.

ρ_t = probability that there is a correction (crash or rally) at time t.

$\kappa_i \in (-\infty, \infty)$ = the size of the i^{th} corrective jump relative to the distance to the normal price. We refer to it as the "crash factor".

η_i = probability that, when there is a correction, it is of size κ_i.

$\bar{K} \equiv \sum_{i=1}^{n} \eta_i \kappa_i$ = expected corrective crash size relative to the distance to the normal price or expected crash factor.

$q_t = \frac{N_t}{p_t}$ is the relative (negative) mispricing of the risky asset.

We introduce the simple stochastic price process with a discrete Poisson process.

$$
\begin{aligned}
& p_{t+1} = p_t \exp\left(\bar{a}_t + \sigma \varepsilon_t\right) \quad \text{with} \quad p_0 > 0 \\
& \text{and} \\
& \bar{a}_t = \begin{cases} \bar{r}_t & \text{with probability } 1-\rho_t \quad \text{with } 0 \le \rho_t < 1 \\ \kappa_i \ln\left(q_t\right) + r_D & \text{with probability } \rho_t \eta_i \quad i = 1,2,\ldots,n \\ & \kappa_i \in \Omega \equiv \left\{\kappa_i \middle| -\infty < \kappa_i < \infty,\ i = 1,2,\ldots,n\right\} \end{cases} \\
& \text{with} \\
& q_t = \frac{N_t}{p_t} \quad \text{and} \quad \sum_{i=1}^{n} \eta_i = 1 \quad 0 < \eta_i < 1 \text{ and } \bar{K} = \sum_{i=1}^{n} \eta_i \kappa_i \\
& N_t = p_0 \exp\left(r_N t\right)
\end{aligned}
\tag{1}
$$

The crash factors are assumed independent and constant over time and distributed according to the probability distribution $\Pi \equiv \left\{\eta_i = \Pr[\text{crash amplitude} = \kappa_i] \middle| i = 1,2,\ldots,n\right\}$. Thus, conditional on no crash happening, which holds at each time step with probability $1-\bar{\rho}$, the price p_t follows a geometric random walk with mean return $\bar{r}_t$ on Δt and volatility σ. We assume that σ is constant, although it is not a necessary condition. At each time step, there is a probability $\bar{\rho}$ for a crash/rally to happen with an amplitude that is proportional to the bubble size, or amplitude defined as $\ln(q_t) = \ln\left(\frac{N_t}{p_t}\right)$ where $N_t = p_0 \exp(r_N t)$ and r_N is defined as the long-term average return[4].

For simplicity of exposition, we will often use a single crash factor, $\bar{K}$, and crash probability $\bar{\rho}$.

In the simplest incarnation of the model, the rates r_D and r_N are constant. When we apply our model to real data, we will want to assume that they vary over time and then characterize them as $r_{D,t}$ and $r_{N,t}$. In this case, we will assume that both r_N and r_D vary with time and that r_N varies slowly while r_D varies more rapidly over time. We may also consider r_D as varying about r_N. When they vary over time, we will want $\lim_{t\to\infty} \frac{1}{t}\sum_{\tau=1}^{t}\left(r_{N,\tau} - r_{D,\tau}\right) = 0$.

A positive κ_i with a $q_t < 1$ means the risky asset is in a positive bubble with a potential correction relative to N_t of size κ_i. A negative κ_i with $q_t > 1$ means that the risky asset is in a regime of transient under-valuation, where the price progressively accelerates downward and will eventually rebound in a rally jump of positive size κ_i times the mispricing amplitude to get closer to the normal price process. The price process model defined by (1) holds for positive ($\ln(q_t) < 0$) and negative ($\ln(q_t) > 0$) bubbles. We allow κ_i to have any real value so that we could replace the discrete jump distribution by a continuous one. In general, and in applications to actual price processes, we will assume that there is a separate distribution Π^+ for positive and Π^- for negative bubbles, consistent with empirical observations.

To see clearly what $\bar{a}_t = \kappa_i \ln(q_t) + r_D$ means, suppose $\kappa_i = 1$. Our price process is such that the crash or recovery is instantaneous and occurs at the beginning of the discrete time interval Δt .Then the occurrence of the crash at time t leads to the price going from p_t to the exact value of the normal price $N_t = p_0 \exp(r_N t)$ and continuing on the interval Δt to p_{t+1} at the rate r_D. The price thus changes

[4] We call it the "normal price return". Some may interpret this as a fundamental price return but that is not the specific intention here.

instantaneously with magnitude $\exp\left(\kappa_i \ln\left(q_t\right)\right)$ at time t and continues changing by $\exp\left(r_D\right)$ over the interval. In other words, $p_{t+1} = N_t \exp\left(r_D\right)$. The price N_t thus acts as a reference price to which the price p_t tends to revert intermittently via the crash occurrences.

We assume that the crash probability is independent and constant over time: $E_{t-1}\left[\rho_t\right] = E\left[\rho_t\right] \equiv \bar{\rho}$: We will relax this assumption later and make it dynamic.

We refer to this specification as corresponding to "efficient crashes", in the sense that their amplitudes are proportional to the bubble size $\ln\left(q_t\right)$, as opposed to being independent of the mispricing. Thus, the more the bubble booms above or below the average fundamental process, the larger the next crash or rally, which will thus tend to bring back the price p_t towards N_t, as argued by Fama (1988) in his analysis of the Oct. 1987 crash. As we will show, this also ensures that, notwithstanding the presence of large bubbles, the price process remains co-integrated with the normal price process on the long term.

Our bubble model does not require one large jump to correct to the normal price. Because of the distribution Π, it can be a sequence of small jumps. It can also be a slower long-term correction depending on the evaluation of r_D after a correction commences.

2.3 Rational Expectation (RE) condition and determination of $\overline{r_t}$

We assume now that the expected return $\overline{r_t}$ is determined in accordance with the Rational Expectation condition $E_t\left(\ln\left(\frac{p_{t+1}}{p_t}\right)\right) = r_D \quad \forall t$, which reads

$$\begin{aligned} E_t\left[\ln\left(\frac{p_{t+1}}{p_t}\right)\right] &= \left(1-\bar{\rho}\right)\overline{r_t} + \bar{\rho}\left(\sum_{i=1}^{n} \eta_i \kappa_i\right)\ln\left(q_t\right) + \bar{\rho} r_D \\ &= \left(1-\bar{\rho}\right)\overline{r_t} + \bar{\rho}\bar{K}\ln\left(\frac{N_t}{p_t}\right) + \bar{\rho} r_D \qquad (2) \\ &= r_D \end{aligned}$$

where $\bar{K}$ is the expected crash factor. With the RE equation, the value $\overline{r_t}$ of the expected return of the risk asset is:

$$\overline{r_t} = r_D - \frac{\bar{\rho}\bar{K}\ln(q_t)}{1-\bar{\rho}} \tag{3}$$

If there is never a crash ($\bar{\rho} = 0$), then the expected return of the risk asset is always r_D .

Equation (3) expresses a positive feedback of the price p_t on the return $\overline{r_t}$ that drives the price process before the crash occurs: the larger the price p_t above the fundamental or normal price N_t, the larger the expected return. As seen from equation (3), this positive feedback results from the assumption that the amplitude of the crash, when it will occur in the future, is proportional to the mispricing $\ln(q_t) = \ln\left(\frac{N_t}{p_t}\right)$. Equation (3) not only relates positively the instantaneous $\overline{r_t}$ to the average crash probability $\bar{\rho}$ and to the average crash factor $\bar{K}$, but also to the log-price, $\ln(p_t)$, in excess to the logarithm of the normal price. Thus, the "efficient crash" condition means that the crash sizes are approximately proportional to the amplitude of the bubble so that the price recovers a value close to the normal price after a crash. The crash is an efficient correction to mispricing in that the price will oscillate about the normal price converging in the limit. Note however that, notwithstanding Condition 2, an efficient crash means that $\bar{K}$ can have any value.

In a positive bubble, the larger the expected log-price, the smaller q_t and thus the larger is $\overline{r_t}$ assuming that r_D is constant: there exists a positive feedback of log-price on return, in a way qualitatively like those documented in laboratory experiments (Hüsler et al., 2013) and from realized as well as implied returns during the 2003-2007 financial bubbles preceding the 2008 crisis (Leiss et al., 2015). Because of this positive feedback, during a bubble phase, the price increases/decreases faster than exponentially, since the return increases/decreases (recall that a constant positive return corresponds to an exponential growth). This phenomenon of increasing returns during bubbles has been documented in (Sornette and Zhou, 2006; Kaizoji and Sornette, 2010; Jiang et al., 2010; Woodard et al., 2010; Corsi and Sornette, 2014) among others. Transient super-exponential price growth has been proposed to be one of the hallmarks of financial bubbles (Sornette, 2003; Sornette et al., 2013a).

Quantitatively, the model in (1) is mildly explosive, similarly to (Phillips and Yu, 2010 and Phillips, Wu and Yu, 2009), since the return in a positive bubble increases with only the logarithm of the price, and not as a positive power of the price as for instance in (Corsi and Sornette, 2014). In Section 5, we will introduce super-exponential exploding bubbles by allowing the probability of a crash to be a function of time and to depend upon the mispricing. In that case, we can also generate finite-time singularities (Sornette and Cauwels, 2015) as $\rho_t \to 1$.

In the risk-neutral framework, the Rational Expectation condition is $\tilde{E}_t\left[\ln\left(\frac{N_t}{p_t}\right)\right]=r_f$, where $\tilde{E}$ stands for the risk-neutral expectation operator and r_f is the risk-free rate, which takes the place of r_D in the risk-neural case. Then we have

$$\begin{aligned}\tilde{E}_t\left[\ln\left(\frac{N_t}{p_t}\right)\right] &= (1-\bar{\rho})\bar{r}_t + \bar{\rho}\bar{K}\ln(q_t) + \bar{\rho}r_f \\ &= r_f \\ &= \bar{r}_t + \frac{\bar{\rho}}{1-\bar{\rho}}\bar{K}\ln(q_t)\end{aligned} \tag{4}$$

In the absence of crashes ($\bar{\rho}=0$), $\bar{r}_t$ reduces to the risk-free rate r_f, as it should in risk-neutral world. During a positive bubble, $q_t<1$ with $\bar{K}>0$, $\bar{r}_t$ is significantly larger than r_f, directly as a result of the existence of crashes. Thus, $\bar{r}_t$ remunerates the investors in excess of the risk-free rate in order to compensate them for taking the risk of being invested in the risky asset that is susceptible to crashes. During a negative bubble, $q_t>1$, the rally compensates the investor for being invested in a declining risky asset. Note that, the larger the probability $\bar{\rho}$ of a crash, the larger $|\bar{r}_t|$. Moreover, the crash risk premium $\bar{r}_t - r_f$ is an increasing function of the amplitude $|\ln(q_t)|$ of the bubble over the normal price $N_t = p_0\exp(r_N t)$ as is the rally premium $r_f - \bar{r}_t$.

2.4 Stationarity and co-integration with the normal price

Proposition 2: Given the bubble model defined by (1) with the RE condition defined by (2) and (3), we have

a) $\lim_{t\to\infty}\frac{1}{t}E\left[\ln\left(p_t\right)\right]=r_D$

b) $\lim_{t\to\infty}\frac{1}{t}E\left[\ln\left(q_t\right)\right]=r_N-r_D$

c) $\lim_{t\to\infty}\frac{1}{t}E\left[\overline{r_t}\right]=\frac{\overline{\rho}}{1-\overline{\rho}}\overline{K}\left(r_D-r_N\right)$

d) When $\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=1}^{t}\left(r_{N,\tau}-r_{D,\tau}\right)=0$ and $r_{N,t}$ and $r_{D,t}$ satisfy Condition 3, our bubble model satisfies the efficient crash Condition 2.

Proof in Appendix A.

Then b) shows that p_t and N_t are co-integrated.

We will be concerned with the case when $r_N=r_D$ or when they vary over time with $\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=1}^{t}\left(r_{N,\tau}-r_{D,\tau}\right)=0$. In this way, our model satisfies the first two conditions postulated in the beginning. In real applications, we assume the crash probabilities and sizes to be different for positive and negative bubbles.

To develop an intuitive understanding of the root mechanism behind Proposition 2, Appendix B dissects a simplified deterministic model of periodically collapsing bubbles with efficient crashes, which can be solved exactly. As already mentioned, this exercise shows that the key ingredient is indeed the “efficient crash” condition, namely that the crash amplitudes are proportional to the mispricing difference between the bubble price and the normal price.

3 Estimation of the parameters

To test and apply our model on real data, we need to estimate several parameters including $\overline{\rho},\overline{K},r_N,r_D$ and σ. We first focus on estimating $\overline{K}$, σ, and $\overline{\rho}$. The idea is to separate the geometric random walk from jumps in the historical data. When we do that, we will separate out σ, meaning that a correct estimation of the model makes the "true" σ associated with the underlying geometric random walk component smaller than the apparent σ computed directly from the historical data without awareness that jumps are present.

A promising approach is to use realized variation and bi-power variation. General assumptions for their application include:

1. Independence of jumps and $\overline{r_t}$.
2. General assumptions are that the jump sizes form a normal distribution, but this can be relaxed.
3. Jumps are independent of the log-price. This may be true, but they may be dependent upon mispricing.
4. There is at most one large jump per day.

Huang and Tauchen (2005) design a significance test based on a relative difference for jumps using a parameter $z_{t-h,t}$, called the *z*-test, which converges to a normal distribution as the sampling frequency goes to infinity. The *z*-test is said to perform impressively when computed daily and does an outstanding job of identifying the days when jumps occur. Tauchen and Zhou (2011) suggest that, after filtering out jumps, a more flexible dynamic structure of the underlying jump arrival rate and jump size distribution can be obtained. See also (Ait-Sahalia and Jacod 2012). Much of the work in this area is on intra-day jumps. Anderson, Bollershev, and Diebold (2007) compute a significant jump and prevent possible negative values in computing the difference between the realized variation and the bi-power variation, which is not possible. This provides a means of selecting "significant" jumps daily based on a α % significance level. These generally rely on intraday data to compute the jumps.

We will be working with daily data and estimating the probability of a jump and jump size over an interval of *d*-days with *d* typically between 5 to 15 business days. The choice of *d* depends on the size and frequency of jumps. When more jumps are present, and the frequency is changing, a shorter size interval is used, whereas when jumps are milder, a longer size closer to 15 days is used. The interval size *d* will be projected into the future to determine an estimated probability and jump size on that interval consistent with our bubble model of equation (1). We take a window of typically 5 years and partition it into intervals of *d*-days. For each of these intervals, we will estimate the realized variance (total variation) and the bi-power variation (variation that is not jumps). We will then use these estimates to obtain an average jump size on a *d*-day interval in the given time window of 5 years. The choice of duration of the time window can however vary around 5 years and reflects the desire to have statistics that are relatively invariant.

We follow the basics of Jacquier and Okou (2014) in our design. We used realized variance composed of continuous volatility and with the jump component embedded in the quadratic variation. They design a statistic based upon the studentized relative difference to test for jumps. This is less useful here as we want to obtain the jump size relative to a variation between the asset price and the normal price. Therefore, we want to know when a variation, r_i, can be considered a jump within a specified significance level.

Instead we use the method of Audrino and Hu (2016) to test if r_i is a jump. Whereas they use intraday data, we apply their method to daily data. Let the history from time *t* be divided into intervals of *d* days and let there be *h* such intervals so that the total number of days of history is $n = hd$. The time *t* is the time for which we wish to determine if the next interval of *d* days will contain a jump. We measure the jumps in each of the *h* intervals of *d* days. We use their statistic where the denominator contains the spot volatility and *k* is taken to be 60[6] days and compute $L_{t,i}$ for each r_i in each of the prior 60 days.

$$L_{t,i} = \frac{r_i}{\sqrt{\frac{1}{k-1}\sum_{j=1}^{k-1}\left|r_{t-j}\right|\left|r_{t-j+1}\right|}} \tag{5}$$

Then $\left|L_{t,i}\right|$ converges to a Gumbel distribution as the sampling frequency tends to zero or, as in Audrino and Hu (2016) we have $\max_{t,i} \frac{\left|L_{t,i}\right| - C_d}{S_d} \to G$.

Therefore r_i is taken to be a jump if

$$\frac{\left|L_{t,i}\right| - C_d}{S_d} > \beta^* \tag{6}$$

where *d* is the number of days in an interval and S_d, C_d, and β^* are parameters from a standard Gumbel distribution: $S_d = \sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\log(d)}}$, $C_d = \sqrt{\frac{\pi}{2}}\left[\sqrt{2\log(d)} - \frac{\log(\pi)+\log(\log(d))}{2\sqrt{2\log(d)}}\right]$,

[6] We use 60 days based on testing giving reasonable results.

and the significance level is $1-\exp\left(-\beta^*\right)$.

We say that r_i is a jump if (6) is satisfied and define

$$I_{t,i}=\begin{cases}1 \text{ if } \dfrac{\left|L_{t,i}\right|-C_d}{S_d}>\beta^* \\ 0 \text{ otherwise}\end{cases} \tag{7}$$

We define jumps relative to *t* and the *h* intervals of size *d*-days. We define the indices for the *lth* interval as $ID_l=\left\{i\left|t-ld+1\le i\le t-\left(l-1\right)d \quad \text{for} \quad l=1,2,\ldots,h\right.\right\}$ and so, as mentioned above, we divide the *n* days into *h* intervals of *d* days. We associate a value $q_l=\dfrac{N_l}{p_l}$ to an interval where $q_l\equiv q_\tau$ for $\tau=t-ld-1$. That is, q_l corresponds to the value of q_τ for the time τ prior to the beginning of the *lth* interval. That is the point by which we determine the asset price relative to the normal price to decide if the jumps in the next interval are for a positive or a negative bubble.

We define positive (*JP*) and negative (*JN*) bubble jumps by interval as:

$$\begin{aligned}
JP_l&=\sum_{i\in ID_l} r_i I_{t,i} \quad \text{if} \quad \log\left(q_l\right)<-\delta \quad \text{for some} \quad \delta>0\\
&\text{and} \quad JP_l^2=\sum_{i\in ID_l} r_i^2 I_{t,i}\\
JN_l&=\sum_{i\in ID_l} r_i I_{t,i} \quad \text{if} \quad \log\left(q_l\right)>\delta\\
&\text{and} \quad JN_l^2=\sum_{i\in ID_l} r_i^2 I_{t,i}\\
JJ_l&=\sum_{i\in ID_l} r_i\left(1-I_{t,i}\right)+\sum_{i\in IQ_l} r_i I_{t,i}\\
&\text{and} \quad JJ_l^2=\sum_{i\in ID_l} r_i^2\left(1-I_{t,i}\right)+\sum_{i\in IQ_l} r_i^2 I_{t,i}\\
&\text{with} \quad IQ_t=\begin{cases}ID_t \text{ if } \left|\log\left(q_l\right)\right|\le\delta\\ \varnothing \text{ if } \left|\log\left(q_l\right)\right|>\delta\end{cases}
\end{aligned} \tag{8}$$

Thus, for each interval, we have associated the total jumps for a positive bubble, a negative bubble, and no bubble. The parameter δ ensures that a jump is not too close to the normal price.

Then the crash amplitude for the *lth* interval for a positive bubble is $\kappa_l^+ = \frac{JP_l}{\ln(q_l)}$ and, for negative bubbles, it is $\kappa_l^- = \frac{JN_l}{\ln(q_l)}$ while the continuous component of the quadratic variation is $\frac{JJ_l^2}{d}$ so that the average σ (used in equation (1)) for a *d*-days interval is $\sigma = \sqrt{\frac{1}{h}\sum_{l=1}^{h} JJ_l^2}$. We have that $\bar{K}^+$ and $\bar{K}^-$ are computed from the average of the κ_l^+ and κ_l^- over those *d*-days crashes or rallies that occur. The probability of a crash or rally is taken from the counts over when they do occur.

We obtain $\sigma, \bar{\rho}, \bar{K}$ separating continuous volatility from jumps defined over *d*-days intervals.

The crucial issues in applying the method is in testing the convergence and selecting the tuning parameters including:

1. *h*: The number of intervals.
2. *d*: the number of days in an interval for estimating the jump size and frequency per interval.
3. α : the significance level $1-\exp\left(-\beta^*\right)$.
4. δ : The tolerance for measuring closeness to the normal price.
5. *k*: the number of days to compute the spot volatility.

Variations in the method are possible but the results obtained below with this parameterization are promising. The most sensitive parameter among the five is the *d* days. A little experimentation on the historical data rapidly determines an excellent value for *d*.

We initially assume that $\sigma, \bar{\rho}, \bar{K}$ are independent of the mispricing with the exception that, in practice, we have separate $\bar{\rho}, \bar{K}$ for positive and negative bubbles. In Section 5, we extend the model by assuming that ρ is a function of the mispricing $\ln(q)$ and/or $\overline{r_t}$ resulting in even stronger super-exponential acceleration and finite time singularities.

We estimate the normal price rate r_N by calibrating a pure exponential price dynamic over a large time window. Ideally, that time window is prior to the beginning of the bubble. It is a window of time when the stochastics of the price process are relatively stable. In experiments, we have generally used 5 to 15 years. This embodies the longer-term price process. The rate r_D is the rate for the short-term component of the price. In experiments, we have estimated it over a window of time prior to the current time, t_2 , for a period of 0.5 to 3 years. We do not index these rates by time here for ease of exposition but in practice estimate them at every *d*-days interval. In a simplified version of the bubble model, we may take $r_N = r_D$.

Demos and Sornette (2016) show for the LPPLS method that the start of the bubble is much less difficult to measure than the end. We observed in experiments that the optimal investment is not very sensitive to the normal price estimation and therefore the start of the bubble.

4 Optimal Investment with the Bubble Model

4.1 Position of the problem

The optimal investment problem is concerned with allocating a portfolio of assets today with the expectation of obtaining "optimal results" in the future. The words "optimal results" can be defined in several ways. For our purposes, we define "optimal results" as the optimal growth of our portfolio. This is also called the method of Kelly (MacLean et. al. 2010a, MacLean and Ziemba. 2010, Thorp 2006, 2010) or maximizing the expected log of wealth. The original work is given in Kelly (1956) with further discussion in Breiman (1961)[7].

Much has been written and researched on the method since Merton (1971). See the compendium of papers in (MacLean et. al. 2010a). Research on the method continues in the extensive work done on fractional Kelly strategies in continuous time with jump-diffusion processes (Davis and Lleo 2013a, 2015a, and 2015b). The fractional Kelly strategy means to only put a fraction of your wealth into the strategy, which makes the results less volatile but may not be optimal in the long-run. Their work is developed in the

[7] Both papers are essentially about gambling and both can also be found in MacLean et. al (2010a). Kelly had originally titled the paper "Information Theory and Gambling" but this was nixed by executives as they thought it would reflect poorly on Bell labs.

context of "risk-sensitive" stochastic control, which means to put the control of the volatility of the portfolio in the objective function. This can be thought of as a risk-averse utility function (Ziemba, 2003). A simple example of a class of risk-averse utilities as a function of wealth w is the constant relative risk aversion (CRRA) class $\frac{w^{1-\gamma}}{1-\gamma} \quad \gamma > 0 \quad = \ln(w)$ as $\gamma \to 1$.

What we do different from Davis and Lleo is that we combine our bubble model with a discrete time Kelly process whereas their jumps are independent of a bubble model and their model is in continuous time.

The impact of bubbles on optimal investment has most to do with the large losses that leveraged investors experience in surfing bubbles resulting in crashes; or the missed opportunities in rallies. We seek to mitigate the losses and advantage the rallies while maintaining the positive properties of Kelly of maximizing the expected log of wealth. We put our bubble model in the context of Kelly because it is illuminating in explaining the process of mitigating crashes and advantaging rallies. For simplicity, we do this for one asset plus a risk-free asset because it provides clarity and easy tests of the methodology.

We do not preclude other methods of optimal investment such as maximizing a different objective in the context of our bubble model with other risk constraints[8]. We investigate that in subsequent work.

4.2 Formalization and solution

Let λ_t be the fraction of wealth W_t allocated to the risky asset in time t and $1-\lambda_t$ the allocation to the risk-free asset with return r_f. Then

$$W_{t+1} = \left(\lambda_t \exp\left(\overline{a}_t + \sigma\varepsilon_t\right) + \left(1-\lambda_t\right)\exp\left(r_f\right)\right)W_t \tag{9}$$

where $\overline{a}_t$ has been defined in (1). We wish to determine $\max_{\lambda_t} E_t\left[\ln\left(\frac{W_{t+1}}{W_t}\right)\right] \equiv L\left(\lambda_t^*\right)$.

where E_t is the expectation conditional on the information up to time t.

For simplicity, we use ρ in place of $\overline{\rho}$ or ρ_t in the following and use the expected crash factor $\overline{\mathrm{K}}$ instead of the distribution of crash factors with probabilities $\rho\eta_i$.

[8] There is a debate on whether to include risks in the objective or as separate risk constraints. One of the authors tends toward the second method and explains why in (Kreuser, 2014).

Define the shifted lognormal process $Y(s_t,\lambda_t)\equiv \exp(r_f)+\lambda_t(\exp(s_t+\sigma\varepsilon)-\exp(r_f))$. Then we have

$$
\begin{aligned}
E_t\left(Y(s_t,\lambda_t)\right) &= \exp(r_f)+\lambda_t\left(\exp\left(s_t+\frac{\sigma^2}{2}\right)-\exp(r_f)\right)\\
Var\left(Y(s_t,\lambda_t)\right) &= \lambda_t^2\left[\exp\left(s_t+\frac{\sigma^2}{2}\right)\right]^2\left(\exp(\sigma^2)-1\right)
\end{aligned}
\qquad (10)
$$

We use the following expression (Elton and Gruber, 1974) valid for the log-normally distributed random variable Y :

$$
\begin{aligned}
E_t\left[\ln(Y)\right] &= \ln\left(E_t(Y)\right)-\frac{1}{2}\ln\left(1+\frac{Var(Y)}{\left(E_t[Y]\right)^2}\right)\\
&= \ln\left(E_t(Y)\right)-\frac{1}{2}\ln\left(\left(E_t[Y]\right)^2+Var(Y)\right)+\frac{1}{2}\ln\left(\left(E_t[Y]\right)^2\right)
\end{aligned}
\qquad (11)
$$

Then the expression for the expected log of wealth (Kelly) follows as:

$$
L(\lambda_t)\ \equiv E_t\left[\ln\left(\frac{W_{t+1}}{W_t}\right)\right]=(1-\rho)E_t\left[\ln\left(Y(\overline{r}_t,\lambda_t)\right)\right]+\rho E_t\left[\ln\left(Y\left(\overline{\mathrm{K}}\ln(q_t)+r_D,\lambda_t\right)\right)\right] \qquad (12)
$$

We substitute (10) into (11) and use that in (12) to get an explicit expression without the expectation. It is the function $L(\lambda_t)$ in (12) that we wish to maximize over λ_t .

The logarithms may not be defined as real numbers for certain values of λ_t with non-positive log arguments. We constrain the domain in the following. Define $\Delta_{i,t}, i=1,2$ and λ_t^L, λ_t^U by:

$$
\begin{aligned}
\Delta_{1,t} &\equiv \exp(r_f)-\exp\left(\kappa_i\ln(q_t)+r_D+\frac{\sigma^2}{2}\right)\\
\Delta_{2,t} &\equiv \exp(r_f)-\exp\left(\overline{r}_t+\frac{\sigma^2}{2}\right)\\
\lambda_t^U-\varepsilon &= \underset{\Delta_{i,t}>0}{\mathrm{Min}}\left[\frac{\exp(r_f)}{\Delta_{i,t}}\right] \quad \text{or } =\infty \quad \text{if no } \Delta_{i,t}>0\ \exists\varepsilon>0\\
\lambda_t^L+\varepsilon &= \underset{\Delta_{i,t}<0}{\mathrm{Max}}\left[\frac{\exp(r_f)}{\Delta_{i,t}}\right] \quad \text{or } =-\infty \quad \text{if no } \Delta_{i,t}<0\ \exists\varepsilon>0
\end{aligned}
\qquad (13)
$$

The above relations express the condition that the log arguments are bounded away from zero. This gives us a bound for $\lambda_t \in \left[\lambda_t^L, \lambda_t^U\right] \quad \lambda_t^L \le 0 \le \lambda_t^U$. We will show that we have a good starting point in $\lambda = 0$ and maximization on λ will keep us away from the boundary where $\ln\left(E_t\left(Y_t\right)\right) = -\infty$.

Then, the first term, $L_1\left(\lambda_t\right) \equiv E_t\left[\ln\left(Y\left(\overline{r}_t, \lambda_t\right)\right)\right]$ in expression (12) can be written as

$$
\begin{aligned}
L_1\left(\lambda_t\right) &= 2\ln\left[e^{r_f}\left(1+\lambda_t\left(e^{z_t}-1\right)\right)\right] - \frac{1}{2}\ln\left(e^{2r_f}\left[\left(1+\lambda_t\left(e^{z_t}-1\right)\right)^2 + \lambda_t^2 e^{2z_t}\left(e^{\sigma^2}-1\right)\right]\right) \\
&= r_f + 2\ln\left[1+\lambda_t\left(e^{z_t}-1\right)\right] - \frac{1}{2}\ln\left(\left(1+\lambda_t\left(e^{z_t}-1\right)\right)^2 + \lambda_t^2 e^{2z_t}\left(e^{\sigma^2}-1\right)\right) \qquad (14)
\end{aligned}
$$

with $z_t = \overline{r}_t - r_f + \dfrac{\sigma^2}{2}$

If we assume a first-order approximation for the exponential and the log, we get:

$$
L_1\left(\lambda_t\right) \approx r_f + \lambda_t z - \frac{1}{2}\lambda_t^2\left[z^2 + \left(1+2z\right)\sigma^2\right] \qquad (15)
$$

Then the approximation is concave if $z^2 + \left(1+2z\right)\sigma^2 > 0$. This convex function has a minimum value of $\sigma^2 - \sigma^4$ at $z = -\sigma^2$. Therefore, $L_1\left(\lambda_t\right)$ is concave if $\sigma^2 < 1$. Assuming this to be the case, then we obtain a value of λ_t of $\dfrac{z}{\sigma^2 + z^2 + 2z\sigma^2}$. Combining both terms in $L\left(\lambda_t\right)$ we get:

$$
\begin{aligned}
\lambda_t^* &\approx \frac{\left(1-\rho\right)\left(\overline{r} - r_f\right) + \rho\left(\overline{\mathrm{K}}\ln\left(q_t\right) + r_D - r_f\right) + \dfrac{\sigma^2}{2}}{\sigma^2 + \left(1-\rho\right)\left(\overline{r} - r_f + \dfrac{\sigma^2}{2}\right)^2 + \rho\left(\overline{\mathrm{K}}\ln\left(q_t\right) + r_D - r_f + \dfrac{\sigma^2}{2}\right)^2 + H} \\
&= \frac{r_D - r_f + \dfrac{\sigma^2}{2}}{\sigma^2 + \left(1-\rho\right)\left(\overline{r} - r_f + \dfrac{\sigma^2}{2}\right)^2 + \rho\left(\overline{\mathrm{K}}\ln\left(q_t\right) + r_D - r_f + \dfrac{\sigma^2}{2}\right)^2 + H}
\end{aligned}
$$

with

$$
H = 2\left(1-\rho\right)\sigma^2\left(\overline{r} - r_f + \frac{\sigma^2}{2}\right) + 2\rho\sigma^2\left(\overline{\mathrm{K}}\ln\left(q_t\right) + r_D - r_f + \frac{\sigma^2}{2}\right) \qquad (16)
$$

We could resort to estimating $L\left(\lambda_t\right)$ via a Taylor expansion as in (Levy and Markowitz, 1979). Rather, we may optimize it over a region on which it is concave. Very powerful nonlinear optimizers[9] can then be applied.

Proposition 3: $L(\lambda)$ is defined and there exits λ_t^A and λ_t^B such that it is a strictly concave function of $\lambda \in \Omega$ with $\Omega = \left\{\lambda \middle| \lambda_t^A \le \lambda \le \lambda_t^B \text{ with } \lambda_t^L \le \lambda_t^A < 0 < \lambda_t^B \le \lambda_t^U\right\}$ provided $\sigma > 0$ or if $\rho \neq 0$ and either $\bar{r} - r_f + \frac{\sigma^2}{2} \neq 0$ or $\bar{\mathrm{K}} \ln\left(q_t\right) + r_D - r_f + \frac{\sigma^2}{2} \neq 0$.

Proof: in appendix C.

The proof of proposition 3 shows that $L(0) = r_f$ and that $L(\lambda)$ is strictly concave around $\lambda = 0$ under mild conditions. Furthermore, $\frac{\partial L}{\partial \lambda} = (1-\rho)\bar{r} + \rho\left(\bar{\mathrm{K}} \ln\left(q_t\right) + r_D\right) - r_f + \frac{\sigma^2}{2}$ indicating the local marginal directional change in L. If we control the size of λ, we can apply optimization directly to L.

By introducing the efficient crashes bubble model, we avoid excessive losses that are experienced in the normal Kelly[10] process in crashes and rallies.

In the next proposition, we estimate a refinement of the approximation of λ_t, which also simplifies in the first order to the above equation (16).

Proposition 4: We can approximate an optimal $\arg\max L\left(\lambda_t\right) = \lambda_t^* \in \Omega$ by

---

[9] See for example optimizers linked to GAMS https://www.gams.com/ . Alternatively, we could apply Golden Section search in this one-dimensional search since the function is concave.
[10] See Thorp 2006, 2010 for more on the Kelly criterion.

$$\lambda^* = \frac{\tilde{D}\left(1+\frac{\sigma^2}{2}\right)-1}{(1-\rho)\left(\tilde{A}-1\right)^2+\rho\left(\tilde{B}-1\right)^2+H_2+H_3}$$

$$\approx \frac{r_D - r_f + \frac{\sigma^2}{2}}{\sigma^2+(1-\rho)\left(\overline{r}-r_f\right)^2+\rho\left(\overline{K}\ln\left(q_t\right)+r_D-r_f\right)^2}$$

with

$$\begin{aligned}
\tilde{A} &\equiv \exp\left(\overline{r}-r_f\right)\\
\tilde{B} &\equiv \exp\left(\overline{K}\ln\left(q_t\right)+r_D-r_f\right)\\
\tilde{D} &\equiv (1-\rho)A+\rho B\\
H_2 &= \left(2\left((1-\rho)\tilde{A}^2+\rho\tilde{B}^2\right)-\tilde{D}\right)\sigma^2\\
H_3 &= \left((1-\rho)\tilde{A}^2+\rho\tilde{B}^2\right)\frac{3\sigma^4}{4}
\end{aligned}$$

Proof in Appendix D.

We see from Proposition (4) that an increasing crash probability (ρ) or mispricing (q_t) drives the allocation of the risky asset to zero, as it should.

5 Dynamics

5.1 Dynamic estimation of parameters

Until now, we have assumed that $\overline{K}, \overline{\rho}$, and r_D are constant. For simplicity, we will keep $\overline{K}$ constant notwithstanding that it probably should be a function of q_t. Surely, the crash probability ρ should be a function of q_t as observed empirically in (Sornette and Zhou, 2006; Sornette et al., 2009; 2010; Woodard et al., 2010). The rate r_D will also change over time. If we had a good estimate of r_D and $\overline{r_t}$, we could compute the crash probability ρ dynamically directly from the RE condition equation (3).

In this section, we want to suggest how we might estimate each of $\rho_t, r_D, \overline{r}_t$ and use these estimates to obtain the best optimal λ_t^*. We suggest methods for estimating each independently and then together through the RE condition.

5.2 Parameterization of the probability function

Evans (2003) states that, 'According to the rational bubble theory, as prices overshoot their fundamental values, there is an increase in the probability the bubble will burst. In turn, the possibility of financial loss increases the risk associated with the ownership of bubbling stock, thereby justifying the acceleration of its price'. Rationality here refers to the idea that investors are supposed to know that there is a bubble component in prices, but prices are guided by self-fulfilling predictions causing prices to rise.

Xiong and Ibbotson (2013) study accelerated stock price increases and note that they are strong contributors of a higher probability of reversal reconciling the 2-12 months momentum phenomenon and one-month reversal. This is further elaborated by Ardila et al. (2016) who introduced the "acceleration" factor.

While it is convenient to assume that the distribution Π is independent of the size of mispricing, it is inconsistent with observed behavior, namely that the likelihood and size of a crash increases with the size of the mispricing induced by an accelerating price.

We assume σ is constant. Alternatively, if we allow it to vary, we assume that it is bounded.

Now we assume that the probability of a crash is a function of the mispricing, $\rho(q_t)$, and seek to estimate the functional form.

We have the actual return $r_t \equiv \ln\left(\frac{p_{t+1}}{p_t}\right) = \overline{r}_t + \sigma\varepsilon_t$, and with the RE condition (3) that

$$\begin{aligned} r_t &= \overline{r}_t + \sigma\varepsilon_t \\ &= r_D - \frac{\overline{\rho}\overline{K}\ln(q_t)}{1-\overline{\rho}} + \sigma\varepsilon_t \end{aligned} \tag{17}$$

We assume a parametric form for the probability as a function of the mispricing for a positive bubble ($q < 1$) and for a negative bubble ($q > 1$) of the form:

$$\rho(q) \equiv \begin{cases} \dfrac{1-q^a}{1+b} & b > -1 \end{cases} \quad \begin{array}{l} \text{if } q < 1 \ \Rightarrow a > 0 \\ \text{if } q > 1 \ \Rightarrow a < 0 \end{array} \tag{18}$$

For a positive or negative bubble, we have $0 < q^a \le 1$ and $a\ln(q) \le 0 \quad \forall a$ as given above.

If we have $-1 < b < 0$, we define $\rho(q) \equiv 1$ for $q^a \le -b$. The case $-1 < b < 0$ results in a finite time singularity. This is because when the denominator is <1, the numerator can attain the value of the denominator with finite mispricing and thus in finite time. For a value of $\rho = 1$, $\overline{r_t}$ becomes infinite through the RE equation. However, that never occurs as the crash probability converges to one and the price crashes before $\overline{r_t}$ becomes infinite.

This family of functions provides a wide range of monotone accelerating probability functions associated with the mispricing and accelerating expected returns.

Define the two-parameter function of q

$$\overline{r_t} = R_t^{a,b}(q_t) \tag{19}$$

where

$$\begin{aligned} R_t^{a,b}(q_t) &\equiv r_D - \frac{\rho_t \overline{K} \ln(q_t)}{1-\rho_t} \\ &= r_D - \frac{(1-q_t^a)}{q_t^a + b} \overline{K} \ln(q_t) \end{aligned} \tag{20}$$

where $q_t = \dfrac{N_t}{p_t}$. We drop the subscript *t* and superscript *a,b* and consider *R* as a function of *q*: $R(q)$. The following Proposition summarizes important properties of ρ and *R(q)*.

Proposition 5: For a positive ($a > 0$) or negative ($a < 0$) bubble, with $b > -1, \ 0 < q^a \le 1$, we get the following:

a. $\dfrac{\partial \rho}{\partial q}$ is $\begin{cases} < 0 & \text{for a positive bubble} \\ > 0 & \text{for a negative buble} \end{cases}$. ρ accelerates to one faster for $|a| \uparrow \infty$.

b. $\frac{\partial R}{\partial q} < 0$ for $a \neq 0$ and increases as $|a| \uparrow \infty$.

c. For $-1 < b < 0$, we have $\rho = 1$ for $q = \hat{q} \equiv (-b)^{1/a}$

d. For $-1 < b < 0$, we have $\lim_{q_t \to \hat{q}} \bar{r}_t = \infty$ and $R(\hat{q}) = \infty$, but $p_t \neq \infty$ because the correction occurs first since $\rho_{t-1} = 1$.

Proof:

For a positive or negative bubble, we have $0 < q^a \leq 1$ and $a \ln(q) \leq 0 \quad \forall a$. Then

$$\frac{\partial \rho}{\partial q} = \frac{-aq^{a-1}}{1+b} \tag{21}$$

$$\frac{\partial^2 \rho}{\partial q^2} = \frac{-a(a-1)q^{a-2}}{1+b} \begin{cases} < 0 & \text{if } |a| > 1 \\ \geq 0 & \text{otherwise} \end{cases} \tag{22}$$

$$\frac{\partial R}{\partial q} = \frac{1}{q^a + b}\left[-\frac{1-q^a}{q} + a\ln(q)q^{a-1}\right] + \frac{a\ln(q)q^{a-1}(1-q^a)}{(q^a+b)^2} \tag{23}$$
$$< 0 \quad a \neq 0$$

The results follow directly from these expressions.

QED

Having obtained a parameterized probability function at each time, if $-1 < b < 0$, then we can obtain a crash probability distribution up to the finite time of a crash, t_c, when ρ goes to 1.

5.3 Calibration of the probability function

If we could measure $\bar{r}_t$, we could compute ρ_t directly from equations (19) and (20). Along with the observable prices, we assume that the parameters $r_D, r_N, \bar{K}, q_t$ are known and define:

$$d_t \equiv \ln(p_{t+1}) - \ln(p_t) - r_D + \frac{(1-q_t^a)}{q_t^a + b}\bar{K}\ln(q_t) \tag{24}$$

We propose to calibrate the parameters a and b of the probability function (24) using weighted least squares:

$$
\begin{aligned}
&\underset{a,b}{Min}\; F_{t_2}(a,b) \equiv \frac{1}{2}\sum_{t\in\Omega(t_2)} w_t^2 d_t^2 \\
&\Omega(t_2) \equiv \left\{ t \middle| t_1 < t \le t_2 \quad \ni \left|\ln(q_t)\right| > \beta \text{ and } \ln(q_t) \text{ has the same sign} \right\} \\
&w_t = \left|\ln(q_t)\right| \\
&a > 0 \text{ for a positive bubble and } < 0 \text{ for a negative bubble} \\
&b > -1
\end{aligned}
\tag{25}
$$

We define t_1 as the beginning of a bubble when $\ln(q_t)$ is close to zero and t_2 the time when the probability is being estimated (i.e. "present" time). In practice, Ω consists of those time periods in a bubble where $\left|\ln(q_t)\right|$ is sufficiently large as defined by the parameter β . The reason for β and the weights w_t is that the fit improves as $\left|\ln(q_t)\right|$ gets large, i.e. when a bubble is well underway.

We are estimating $\overline{r}_t$ through the RE equation. We know that it can take a long time to estimate an expected return (Merton, 1980; Ambarish and Siegel, 1996). The later reference shows that the time to get an estimate of the expected return to a given confidence level is a function of $\left(\frac{\sigma}{\overline{r}_t - r_f}\right)^2$. The reason (25) can work is that we have assumed that σ is constant (or at least bounded) whereas $\left|\overline{r}_t\right|$ is accelerating. The return process has normal variates contingent on no crash and the time to estimate it rapidly decreases as $\left|\overline{r}_t\right|$ gets large, which gives us the rational for weighting the observations by $\left|\ln(q_t)\right|^2$ as, the larger it gets, the larger is $\left|\overline{r}_t\right|$. We scale *F* for numerical convenience.

We do the fit over several windows of time. If the fit is "good enough", we assume the price process is in bubble phase and the resulting probability function defines the crash probability locally as a function of the mispricing. The solution (a,b) gives us the probability of a crash at time *t* since $\rho(q_t) = \frac{1-q_t^a}{1+b}$. We use this probability to plug into our Kelly equation given in Proposition 4 to determine the allocation between the risky and riskless assets after reconciling it through the RE condition.

5.4 The discount rate, the normal price rate, and the expected return.

It can happen that a negative bubble occurs when transiently $q < 1$ or a positive bubble when $q > 1$. If a jump related to the bubble is large enough, we can assume that the jump is to a new normal price. We then redefine the start of our new normal price $N_{t_1} = p_{t_1}$. We will investigate this further in a subsequent empirical study.

The discount rate r_D is estimated over a time window less than the window used to calculate the normal price. We estimate them so that we have approximately $\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=1}^{t}\left(r_{N,\tau} - r_{D,\tau}\right) = 0$.

We can get a reasonable estimate directly on the expected return, $\overline{r_t}$, especially when the crash probability, ρ_t, is large as this signals an accelerating return.

5.5 Rationalizing the parameter estimates

We have assumed that we can estimate $\rho_t, \overline{r_t}, r_D$ independently. Having these estimates, we should rationalize them through the RE condition.

We could look at each of $\rho_t, \overline{r_t}, r_D$ to see which can be determined to be most accurate and then use the RE condition to adjust the values appropriately. If we have good estimates of ρ_t and r_D, we could calculate $\overline{r_t}$ through the RE condition.

We observe (for example Fig. 4 in Section 6) that it is often the case that prices plateau before finally crashing. During such a plateau, the value of q_t decreases progressively as the risky asset price remains approximately constant while the normal price increases, steadily catching up and reducing the bubble component. During the plateau, the expected return $\overline{r_t}$ decreases progressively to r_D. This implies that ρ_t decreases though the RE condition, but it does so slowly. In fact, ρ_t is sticky and reflects how long the bubble has lasted and how strongly it has developed. Indeed, the longer and deeper the bubble, the more time it takes for the normal price to catch up. Moreover, there can be an adjustment of the risk-adjusted discount rate r_D to reflect the perceived decreased risk by investors. This (misguided) perception may lead to a compensation of the decrease of $\overline{r_t}$ during the plateau, so that the difference $\overline{r_t} - r_D$ may remain essentially unchanged and therefore, by the RE condition, the crash probability remains approximately unchanged and large. This scenario can happen both in a positive or a negative bubble. The result is that the value of the asset allocation λ_t^* may go to zero or negative prior to a crash or to zero or positive prior to a rally, a behavior our investor would desire.

Once we have good estimates of ρ_t, $\overline{r_t}$, and r_D, we calculate λ_t^* through the approximating equation, or via Golden Section search if one-dimensional, or we could maximize the locally concave and continuous function given in (12) .

6 Examples on historical bubbles

We show here several examples of the application of our method that applies the Kelly criterion in the context of our efficient crashes bubble model. Each of the following examples has two graphs. The first graph consists of the following plots:

1. Optimal efficient portfolio: consisting of an optimal allocation between the asset and the risk-free rate, based upon the solution obtained above (Proposition 4) using the Kelly criterion applied to our model that quantifies risks as a mixture of volatility and jumps.
2. Actual price: The actual price of the risky asset. All values and prices are measured in logs.
3. Normal price: Computed dynamically as explained above so it can change slope over time.
4. BM Kelly portfolio: The classical Kelly allocation between the asset and the risk-free rate that quantifies risks solely based on return volatility. It uses the volatility computed directly from the historical data and not the σ computed in Section 3.
5. 60/40 portfolio: 60% in the asset and 40% in the risk-free rate.

The second graph consists of the following plots:

1. Lambda: The value of the Kelly allocation λ_t^*.
2. Actual price: The actual log of the price of the risky asset.
3. Normal price: Computed as per the above plot.

Rebalancing of the portfolio takes place every d-days as this was the interval size used in the estimation of Section 3. Transaction costs are not considered.

In most cases, the maximum drawdown of our efficient portfolio is significantly smaller than that of the risky asset. Maximum drawdown is measured as the maximum over the entire period.

In all cases, the CAGR[11], Sharpe, and the Calmar Ratio[12] of the efficient portfolio are better than that of the asset.

We limit leverage or shorting to 100% of the portfolio for either the efficient portfolio or the classical Kelly portfolio. Larger values will produce more volatile solutions. Even if these solutions have a better CAGR value, they will likely have a lower Calmar ratio.

Algorithm:

1. Estimate r_N and r_D as in Section 3.
2. Estimate $\overline{\rho}$ and $\overline{K}$ on historical data for a d-days interval separately for positive and for negative bubbles as in Section 3. Also estimate σ as in Section 3.
3. Obtain an estimate of $\overline{r_t}$ and measure its goodness of fit.
4. Estimate the probability function parameters (a,b) on windows of diverse sizes. Constraint $a>0$ or $a<0$ depending on whether it is a positive or negative bubble. Compute the probability ρ_t of a correction on the next d-days interval and measure its goodness of fit.
5. Rationalize $r_D, \overline{r}_t, \rho_t$ through the RE condition as in Section 5.5. If ρ_{t-1} has accelerated in a bubble but q_t does not change much from q_{t-1}, keep it sticky (do not change ρ_t much from ρ_{t-1}) and instead adjust r_D to satisfy the RE equation.
6. Compute the asset allocation λ_t^* using Proposition 4 or an optimization technique.

In all figures, the compound annualized growth rate (CAGR) and the maximum drawdown (Max Drawdown) are given in percent.

Figure 1 shows the typical behavior of the efficient portfolio for the Hong Kong market. The classical Kelly portfolio over-bets and then crashes farther. The efficient portfolio begins to mitigate the crash

[11] The compound annualized growth rate.
[12] The CAGR divided by the maximum drawdown (Young, 1991).

because $r_D - r_f$ becomes negative when the estimated r_D drops below the risk-free rate r_f around 13 Oct 1997. This in turn creates a negative λ_t^* resulting in shorting the asset.

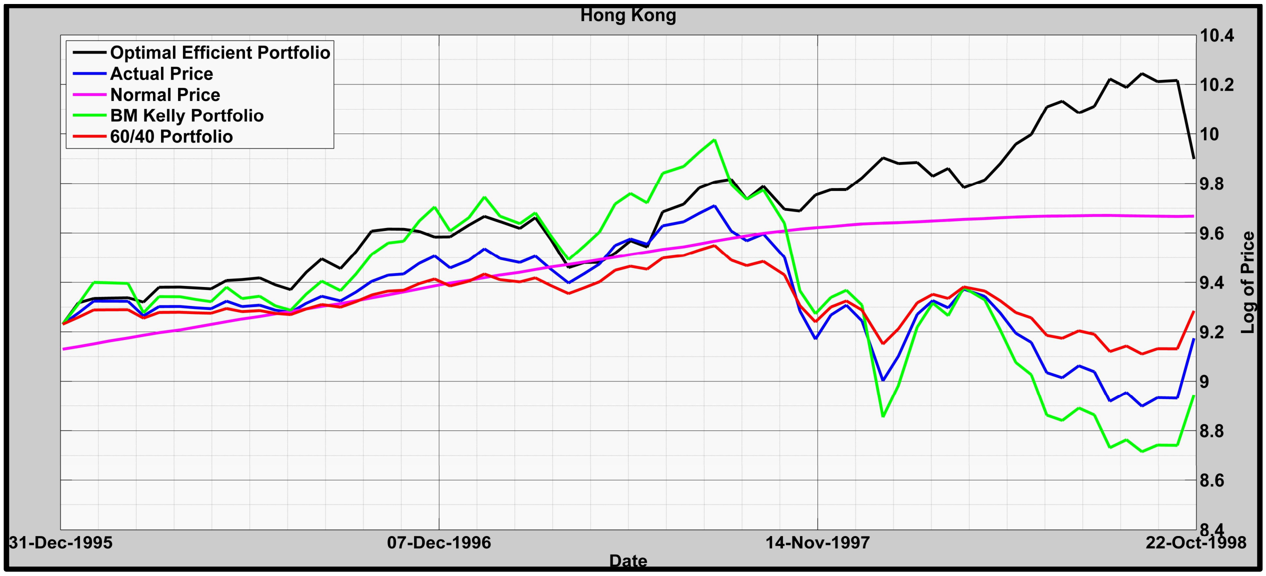
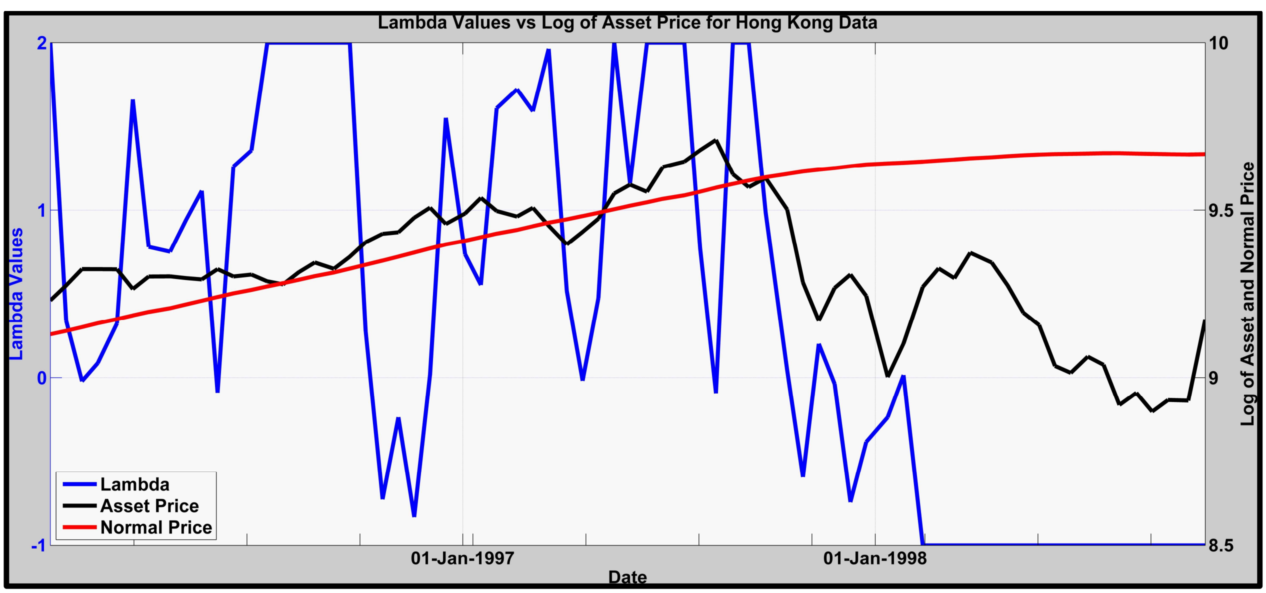


Fig. 1: Hong Kong – 10 Jan 1996 to 20 Oct 1998				
	CAGR	Sharpe	Max Drawdown	Calmar
Actual Price	-2.37	-0.19	56.40	-0.04
Optimal Efficient Portfolio	35.65	0.70	29.16	1.22



Brent Oil in Figure 2 exhibits a similar behavior of the efficient portfolio, except that it reverses the sustained downturn and catches the rally. The classical Kelly is even worse here as it is leveraged to the limit $\lambda_t^* = 2$. The extended drawdown lasts from July 2008 to February 2009. The efficient portfolio recovers as λ_t^* begin dropping from 2 to -1 on 26 September 2008. During that same period, $r_D - r_f$ becomes negative for the same reason as in the Hong Kong case and is the cause for λ_t^* becoming negative. It is not until further into 2009 that the rally size and probability become large enough to push λ_t^* toward plus one. All the statistics of the efficient portfolio are considerably better than that of the actual price with a CAGR more than trebled.

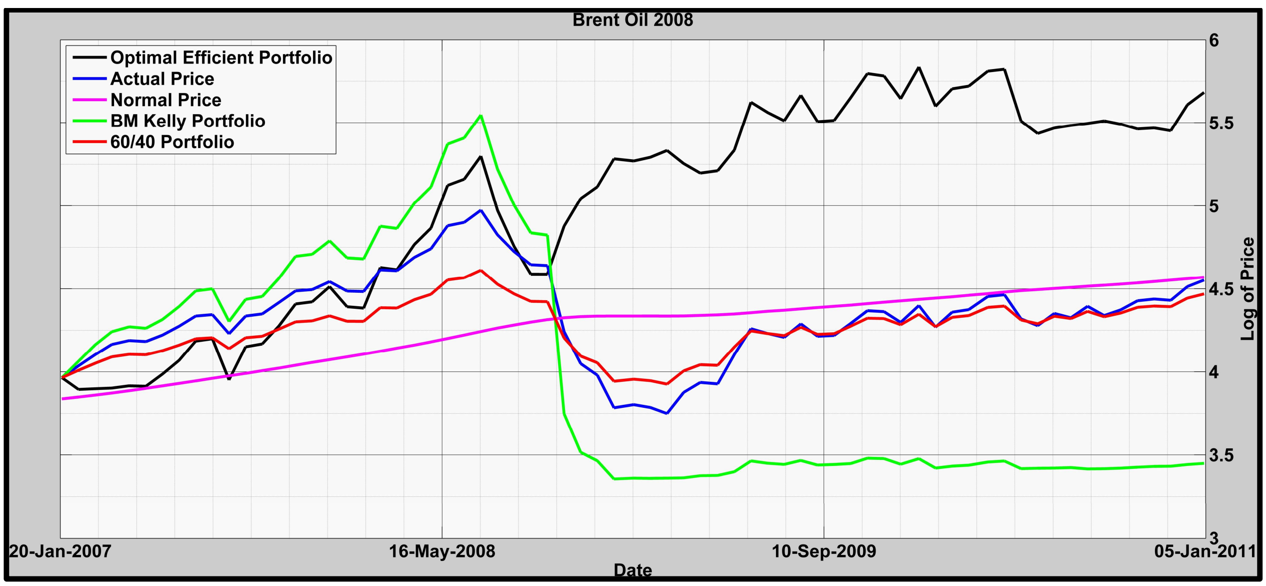
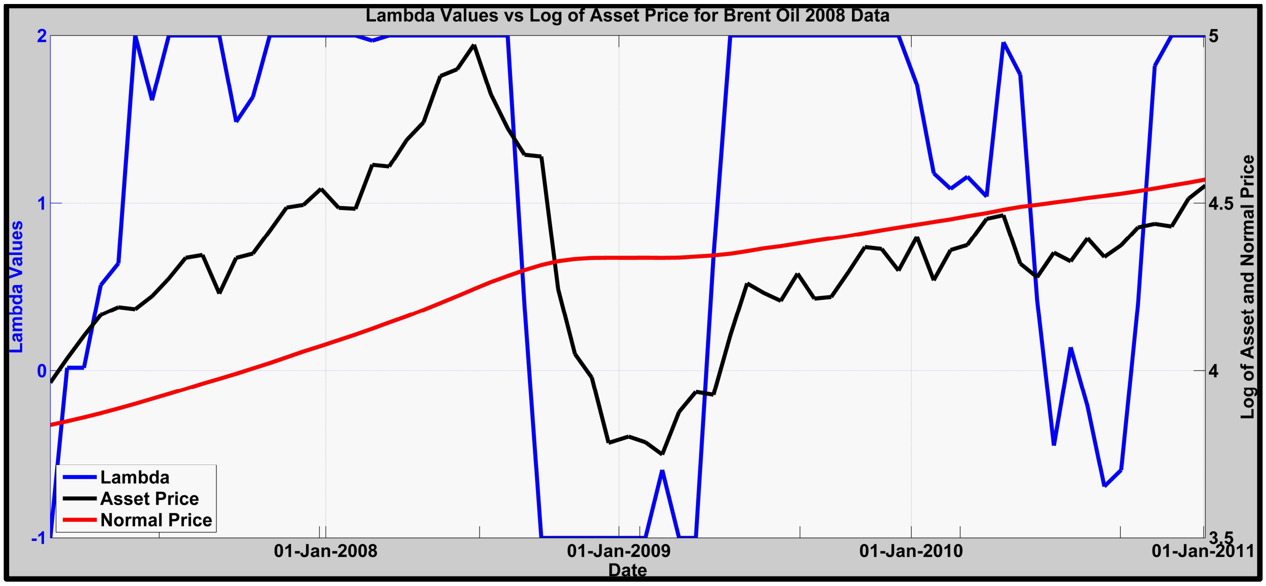


Fig 2: Brent Oil – 4 Jan 2005 to 3 Jan 2011				
	CAGR	Sharpe	Max Drawdown	Calmar
Actual Price	14.69	0.24	70.57	0.21
Optimal Efficient Portfolio	42.94	0.70	50.85	0.84



In the lambda graph above, we have Brent oil peaking on 4-Jul-08. It was on 5-Sep-08 that the value of lambda dropped to almost zero and then on 26-Sep-08 it went to -1 effectively shorting the asset prior to the major downturn. It also began to signal an upturn after that.

Figure 3 shows that the efficient portfolio mitigates the crash in the Dow Jones 1929 very well. As is often the case, the classical Kelly overshoots a lot when leveraged. The classical Kelly is much more leveraged going into the crash than the efficient portfolio. This is because the efficient portfolio shows a high crash probability and large value of $\overline{K}$. The lambda graph shows it dropping over several months prior to the crash and catching some of the rally and subsequent downturn.

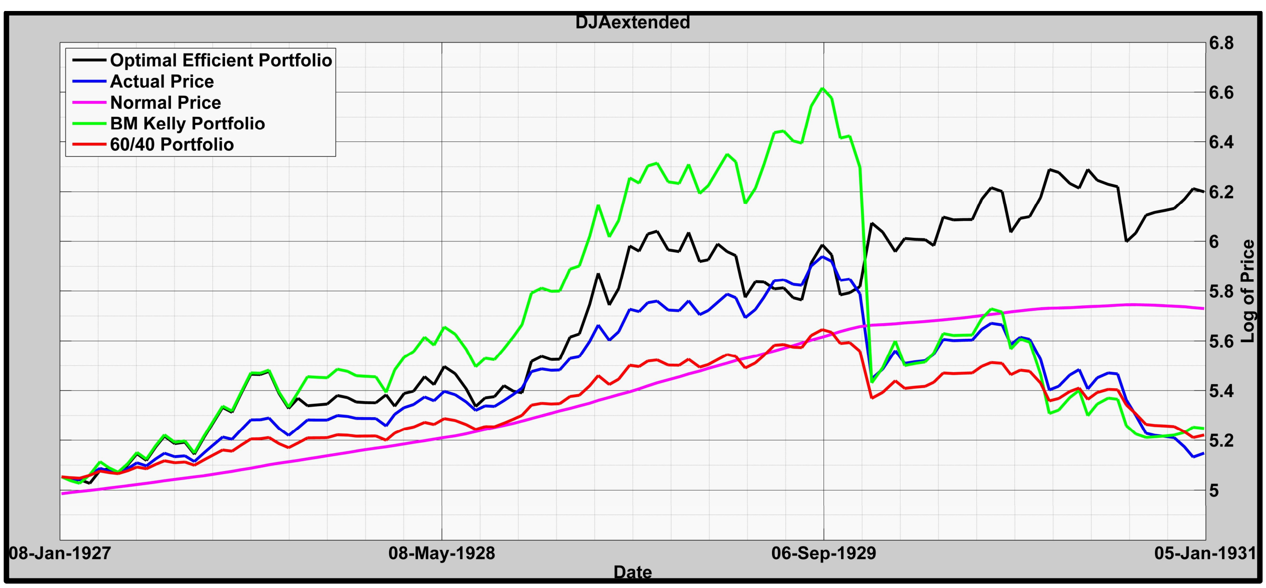


Fig 3: Dow Jones 1929 – 7 Jan 1927 to 3 Jan 1931

	CAGR	Sharpe	Max Drawdown	Calmar
Actual Price	2.37	-0.08	55.37	0.04
Optimal Efficient Portfolio	28.65	0.61	25.26	1.13

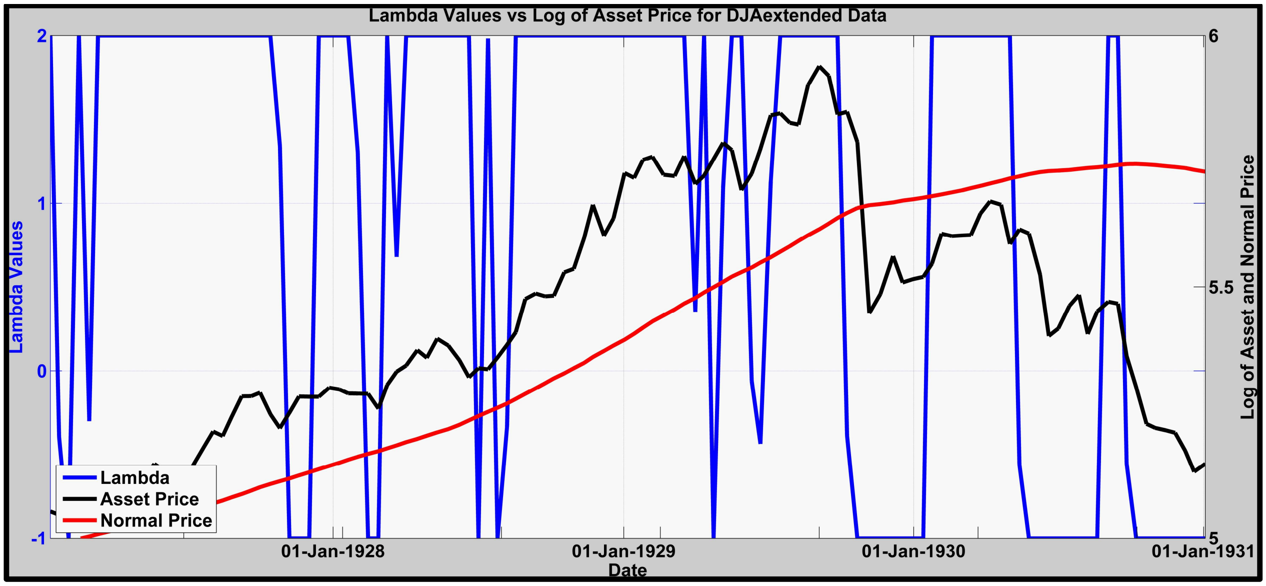


Figure 4 shows for the S&P 500 2007 the efficient portfolio first in sync with the classical Kelly, shooting up when the asset price is shooting down. Interestingly both shoot down a bit at the end when the asset price shoots up. This is again because the value of $r_D - r_f$ becomes negative and stays there throughout the duration of the period. The classical Kelly is similar except it is a little slow to catch the upturn. The window for computing r_D is one year. This is too long to catch the upturn at the end for either the efficient portfolio or the classical Kelly. Therefore, they both turn down a bit despite the asset rallying. The extended drawdown in the asset price begins around October 2007 until March 2009.

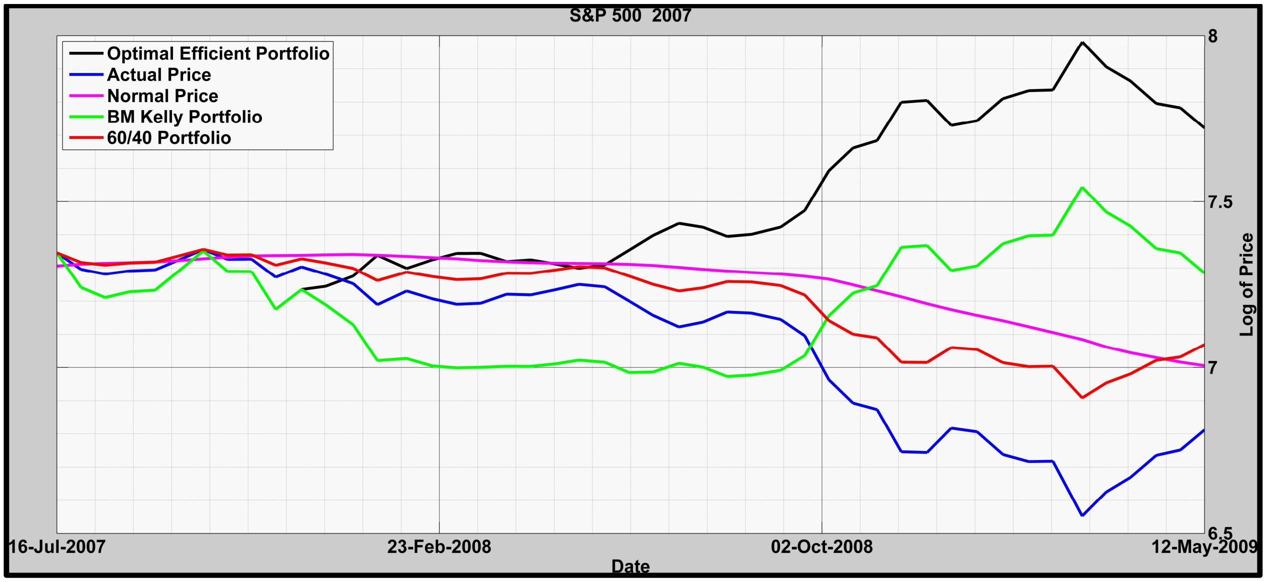
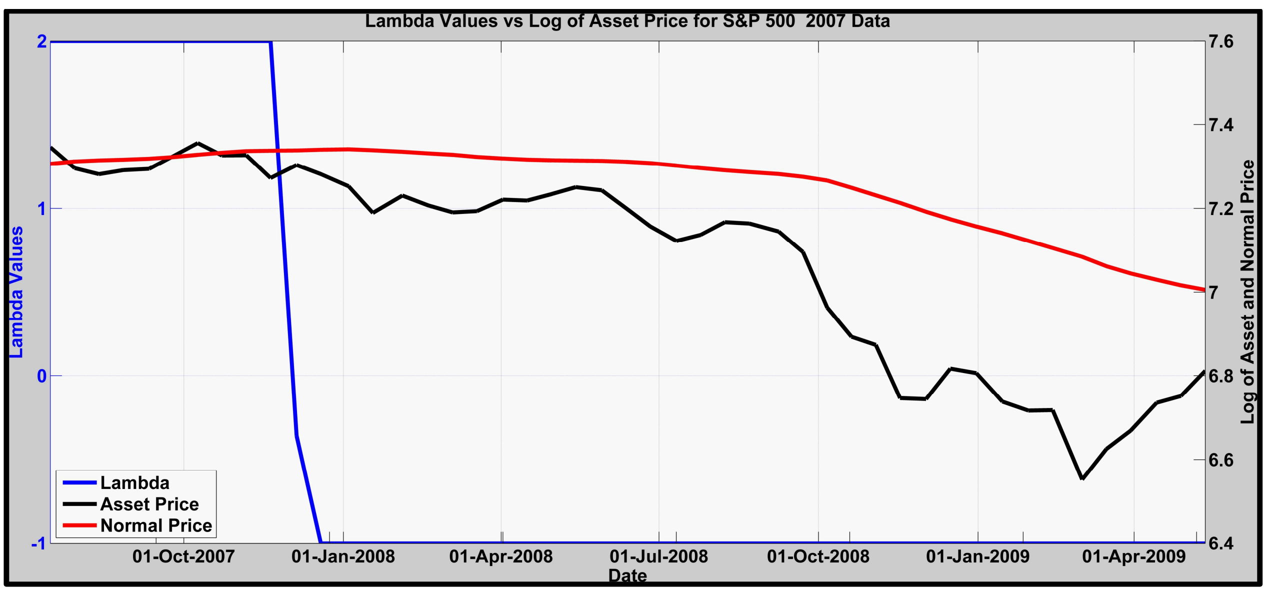


Fig 4: **S&P 500 2007** – 2 Jul 2007 to 12 May 2009				
	CAGR	Sharpe	Max Drawdown	Calmar
Actual Price	-28.49	-1.31	55.22	-0.52
Optimal Efficient Portfolio	20.05	0.59	22.83	0.88



The case of MSCI Russia shown in Figure 5 is a bit different. Both the classical Kelly and the efficient portfolio are leveraged to August 1997. Only Kelly is leveraged a little more. The problem begins in August 1997 when the asset price turns followed by a continuing downturn. But the efficient portfolio can hedge that because the crash probability jumps up along with a substantial increase in the value of $\bar{K}$ and

$|\ln(q_t)|$. The classical Kelly is again too highly leveraged although it is doing better before the downturn. The efficient portfolio continues outperforming on the downturn and over a two year period outperforms by a factor of 10. The result is that the optimal efficient portfolio has a significantly higher CAGR and Calmar than the asset price. Of course, if we had stopped measuring in September 1997, the classical Kelly would have beaten all. But that is just the point of the efficient portfolio to beat out the other methods over crashes and rallies.

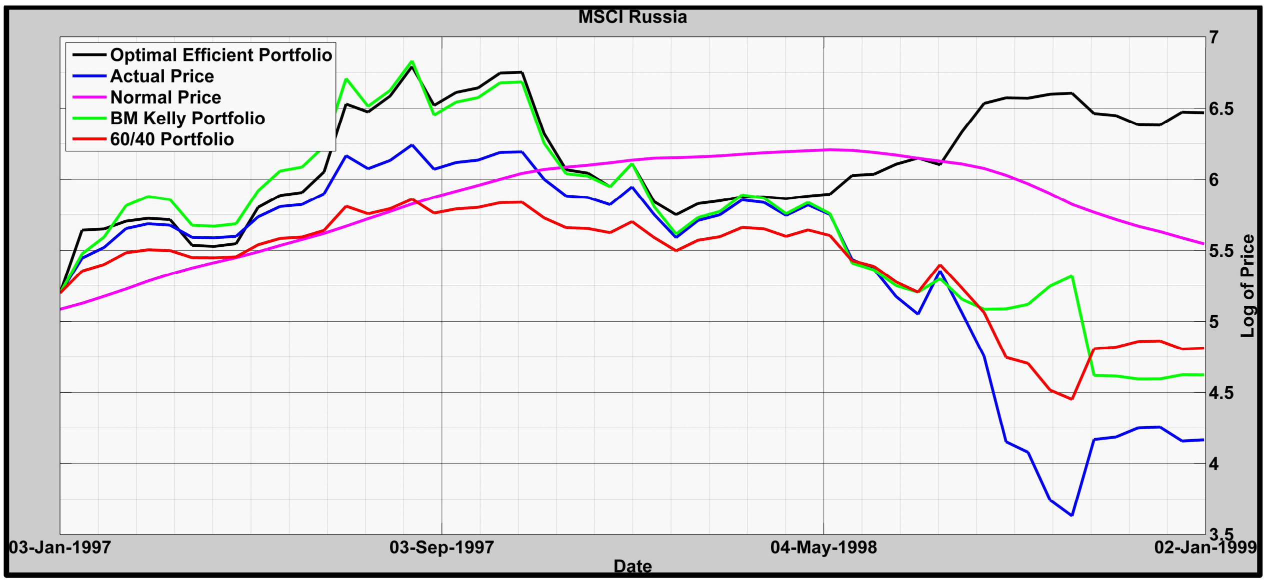


Fig 5: **MSCI Russia** – 2 Jan 1997 to 1 Jan 1999				
	CAGR	Sharpe	Max Drawdown	Calmar
Actual Price	-51.65	-0.60	92.66	-0.56
Optimal Efficient Portfolio	63.51	0.74	64.73	0.98

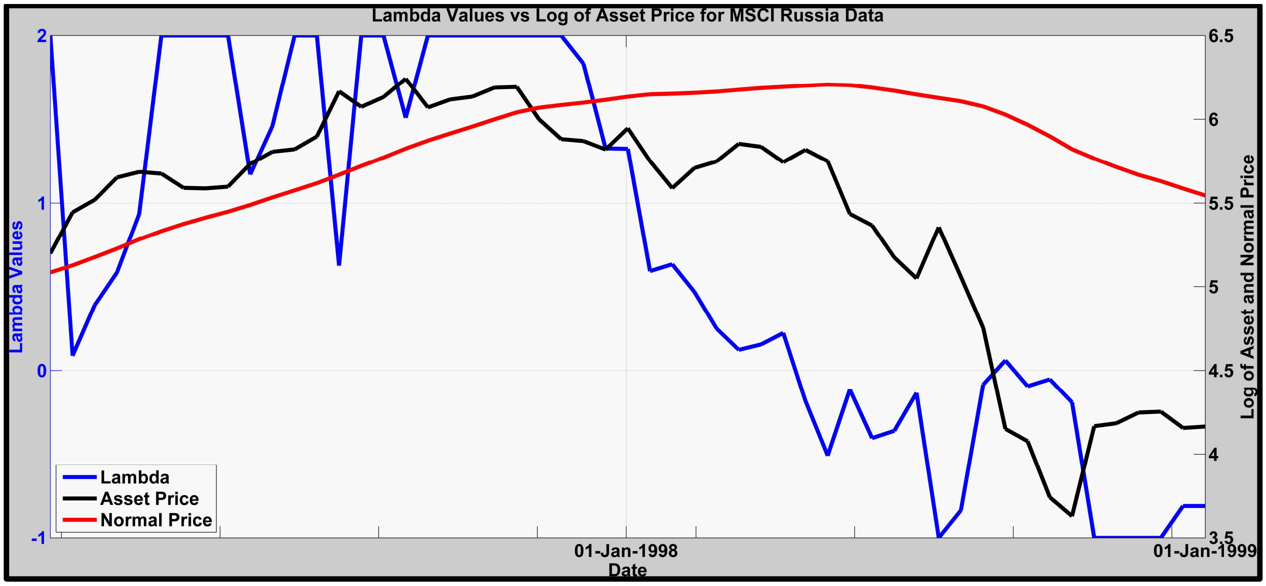


Figure 6 for Gold shows another interesting example. The efficient portfolio is essentially shorting Gold the entire period. One of the results of that is a significantly reduced drawdown and subsequently improved Calmar over the asset price. It is important to note that the performance in this case has much to do with computing the dynamic normal price.

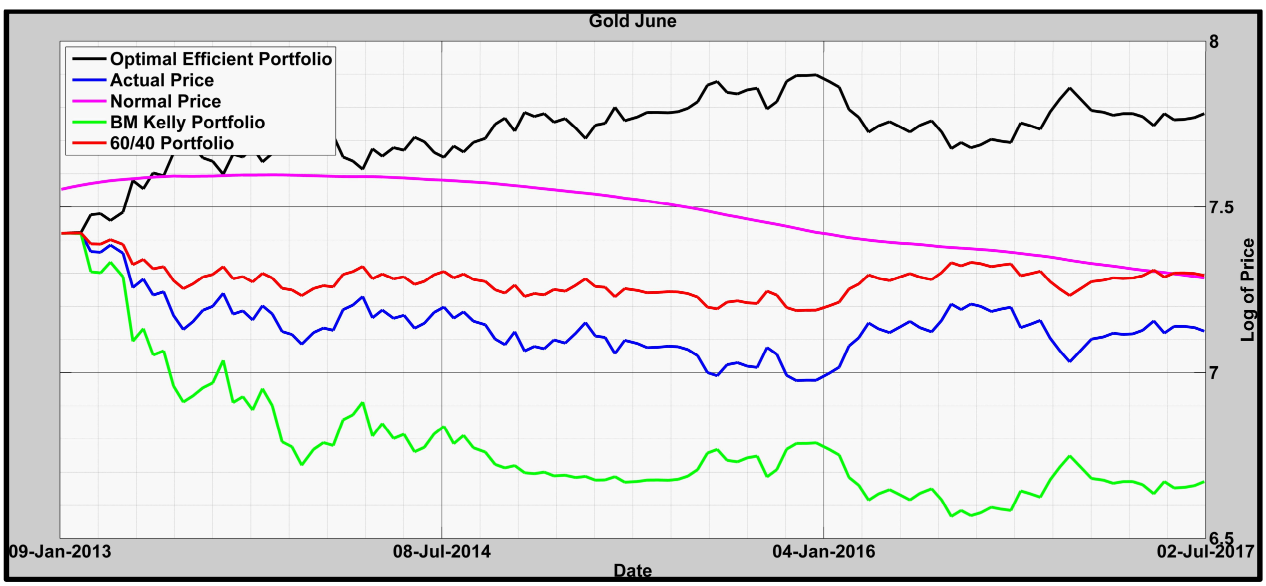


Fig 6: Gold – 11 Jan 2013 to 30 Jun 2017				
	CAGR	Sharpe	Max Drawdown	Calmar
Actual Price	-6.55	-0.54	35.91	-0.18
Optimal Efficient Portfolio	8.02	0.38	19.91	0.40

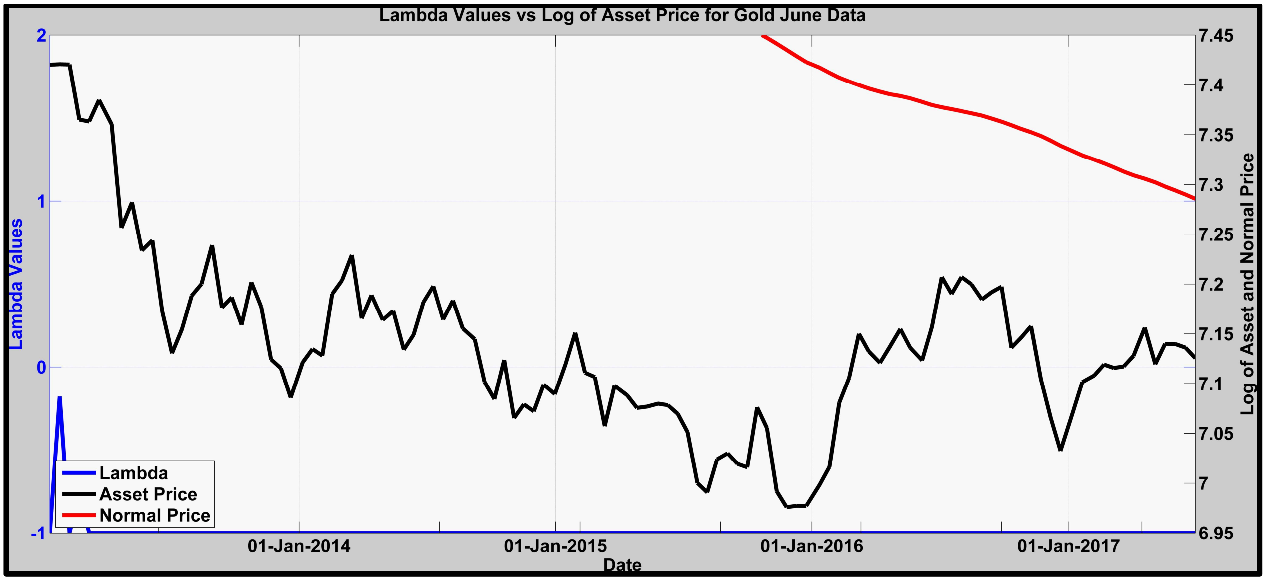


Figure 7 shows the results for the S&P 500 over a 30 year period from 17 May 1988 to 16 May 2018.. In this case, the efficient portfolio is leveraged more than the classical Kelly because the crash probability and size, $\overline{K}$, increase over time. The result is a significant Calmar difference between the classical Kelly and the efficient portfolio. The efficient portfolio has a significantly better CAGR than the asset. The efficient portfolio nicely avoided some corrections and did catch some rallies.

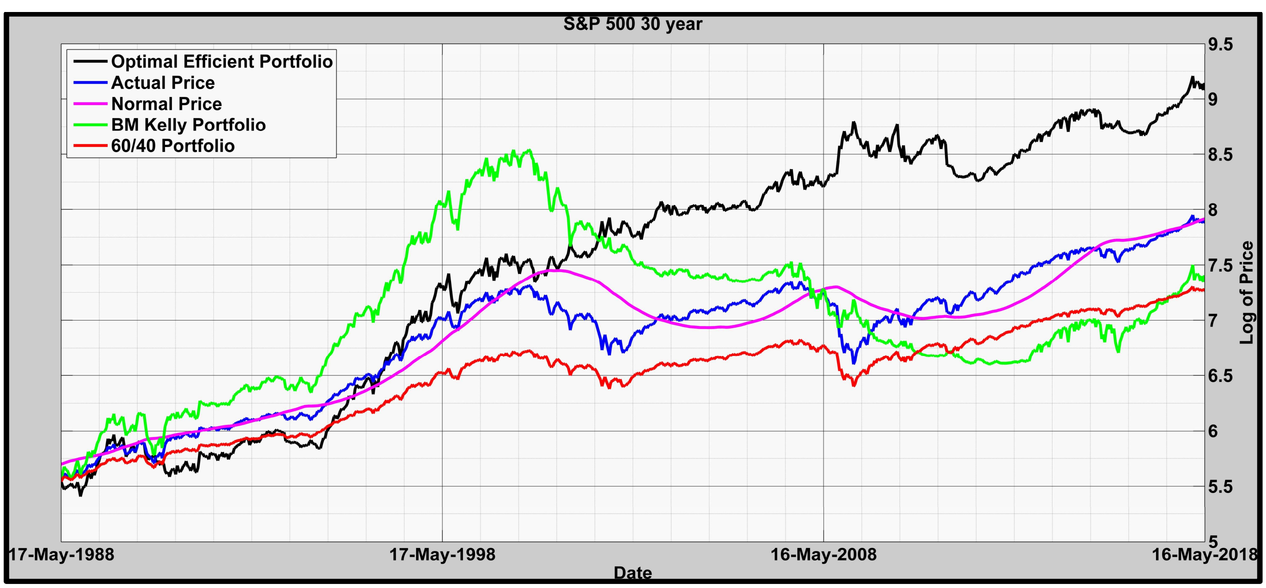
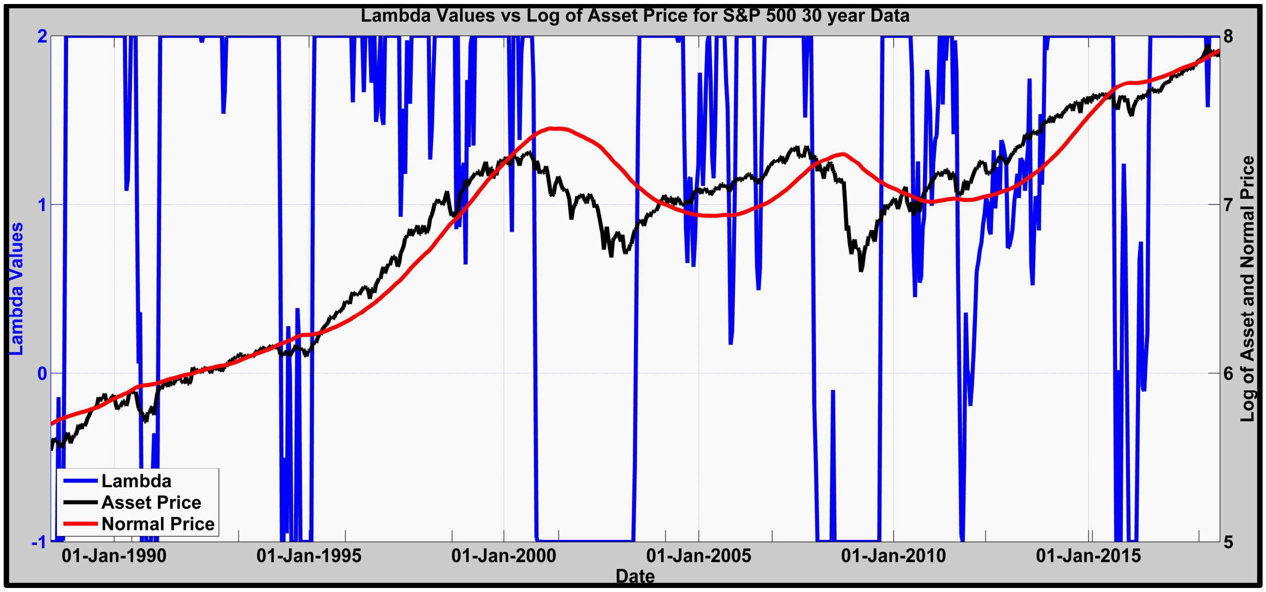


Fig 7: S&P 500 30 years: 17-May-1988 to 16-May-2018.				
	CAGR	Sharpe	Max Drawdown	Calmar
Actual Price	7.89	0.40	52.65	0.15
Optimal Efficient Portfolio	12.00	0.46	41.61	0.29



However, it looks like the efficient portfolio underperforms the asset for most of the first 10 years. Holding the estimation parameters for 30 years is not something we would suggest in practice. We reran the first ten years with a modest change on one of the parameters. The only difference between this run and the 30-year run was in lengthening the estimation window to calculate the parameter r_D from one year to four years. The result is given in Figure 8. There we see the efficient portfolio outperforming and keeping up with the Kelly portfolio in a raising market.

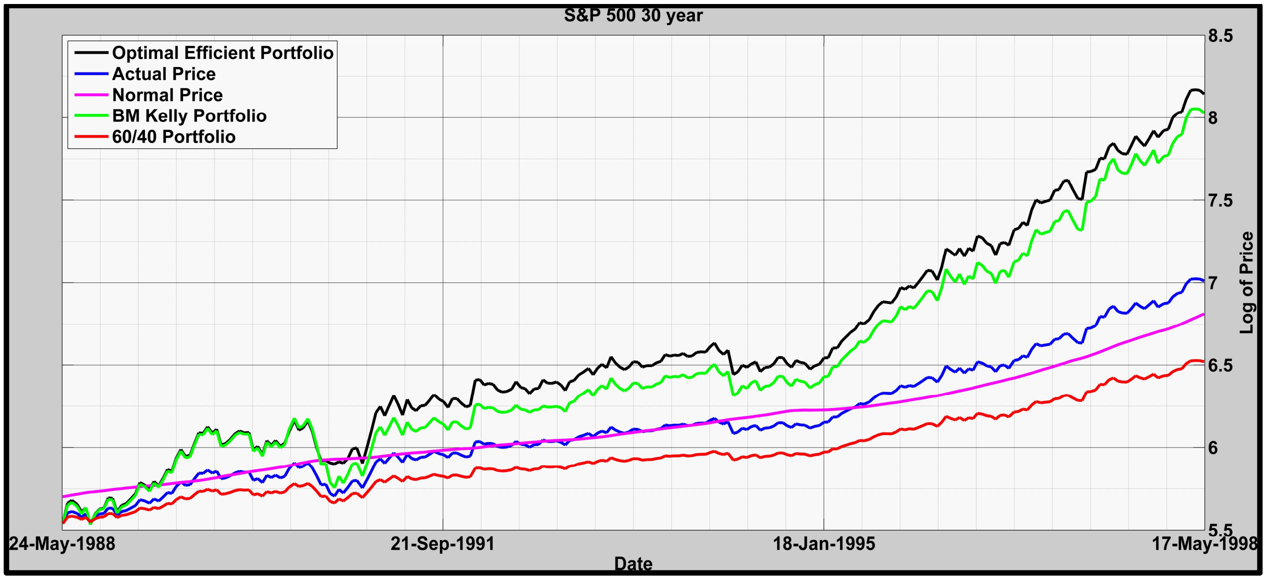


Fig 8: S&P 500 10 years: 16-May-1988 to 15-May-1998.

The following table summarizes the differences between the efficient portfolio and the asset price over the seven analyzed time series. There are no negative values in the table.

Table of Differences: Efficient portfolio value minus asset price value				
Problem	**CAGR**	**Sharpe**	**Drawdown** [1]	**Calmar**
Hong Kong	38.02	0.89	27.24	1.26
Brent Oil	28.25	0.46	19.72	0.63
DJ 1929	26.28	0.69	30.11	1.09
S&P 500 2007	48.54	1.9	32.39	1.40
Russia 1997	63.51	0.74	64.73	0.98
Gold	14.57	0.92	16.00	0.58
S&P 500 30 year	4.11	0.06	11.04	0.14
Average	39.28	0.89	23.49	0.95

Notes:

1. The difference in drawdown is asset price drawdown value minus efficient portfolio drawdown value. In all cases the larger the value the better is the efficient portfolio.

7 Conclusions and further work

We have proposed a rational expectations bubble model with efficient crashes and have shown it to be consistent with many concepts usually invoked in classical bubble models. We have evaluated the bubble model by showing that, in combination with an optimal investment methodology, it can perform well on historical data and on bubbles. Furthermore, it compared favorably to other portfolio methods such as the classical Kelly and a 60/40 portfolio as part of our evaluation.

There are obvious improvements in the methodology that can be made. Most important is the rationalization of the dynamic discount rate, crash probability, and the expected return through the rational expectations equation. We may also consider the calibration of the crash size dependent upon the mispricing.

Clearly, more computational evaluations should be conducted on historical data and additional forward testing of the method should be developed. Transaction costs need also to be added as part of the evaluation.

On the applications side, the methodology can be extended to a multi-asset version. This will include investigating other optimal investment methods than Kelly.

On the theoretical side, we may wish to develop a continuous time version of the bubble model and method. The proposed existence of efficient crashes may provide explanations for some of the existing pricing anomalies in the empirical literature.

Appendix A:

Proposition 2: Given the bubble model defined by (1), with the RE condition defined by (2) and (3), and which satisfies the efficient crash condition, we have

a) $\lim_{t\to\infty}\frac{1}{t}E\left[\ln\left(p_t\right)\right]=r_D$

b) $\lim_{t\to\infty}\frac{1}{t}E\left[\ln\left(q_t\right)\right]=r_N-r_D$

c) $\lim_{t\to\infty}\frac{1}{t}E\left[\overline{r_t}\right]=\frac{\overline{\rho}}{1-\overline{\rho}}\overline{K}\left(r_D-r_N\right)$

d) When $\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=1}^{t}\left(r_{N,\tau}-r_{D,\tau}\right)=0$ and $r_{N,t}$ and $r_{D,t}$ satisfy Condition 3, our bubble model satisfies the efficient crash Condition 2.

Proof:

First, note that these three results are consistent with (3). We assume, without loss of generality that $p_0 = q_0 = 1$.

By (3), we have $E\left[\ln\left(p_t\right)\right] = E\left[\ln\left(p_{t-1}\right)\right] + r_D$, which is the RE condition.

So $E\left[\ln\left(p_t\right)\right] = tr_D$ and **a)** follows.

We have from the definition of q_t that $E\left[\ln\left(q_t\right)\right] = \ln\left(p_0\right) + tr_N - E\left[\ln\left(p_t\right)\right]$

And from (3) (RE condition) we have $E\left[\ln\left(p_t\right)\right] = E\left[\ln\left(p_{t-1}\right)\right] + r_D$

Therefore, we have

$$\begin{aligned} E\left[\ln\left(q_t\right)\right] &= \ln\left(p_0\right) + r_N\left(t-1\right) + r_N - E\left[\ln\left(p_{t-1}\right)\right] - r_D \\ &= E\left[\ln\left(q_{t-1}\right)\right] + r_N - r_D \end{aligned}$$

And so $E\left[\ln\left(q_t\right)\right] = t\left(r_N - r_D\right)$.

The results for **b)** follows.

We have from (4) that $E\left(\overline{r}_t\right) = r_D - \dfrac{\overline{\rho}\overline{K}E\left(\ln\left(q_t\right)\right)}{1-\overline{\rho}}$

And using the result for **b)** gives **c).**

We get d) by letting $B_t = \exp\left(\overline{a}_t - r_N\right)$ and applying Khintchin's proof for the Weak Law of Large Numbers.

QED

Appendix B: Simplified deterministic model of periodically collapsing bubbles with efficient crashes

Let $Y_t := \ln\left(p_t^N\right)$ where p_t^N is the normal price process $p_t^N = p_0 \exp(r_N t)$. We take $p_0 = 1$ with no loss of generality. To simplify, we assume $r_N = r_D$ and present only the case of a positive bubble. However, we stress that the model can equally handle negative bubbles.

We imagine the simplified process decomposed in discrete time intervals of duration T. Let $X_t := \ln\left(p_t\right)$ be the logarithm of the price process. The process starts at time t=0 for which $X_0 := \ln\left(p_0\right) = 0$. For times between 0 to T$^-$, the price grows at the return $\bar{r} > r_N$ as $p_t = p_0 \exp(\bar{r}t)$. At time T, the excess return is $\ln \frac{p_T}{\exp(r_N T)}$. Then, a crash occurs with certainty with amplitude k times this excess return, with $k \leq 1$. This is the specialization of the rule in (1) of our simplified model for the crash amplitude controlled by the mispricing ratio q_t. From T$^+$ to 2T$^-$, the price grows again at the rate $\bar{r} > r_N$. At time 2T, it crashes again with the amplitude $k \ \ln \frac{p_{2T}}{\exp(2 r_N T)}$ and so on.

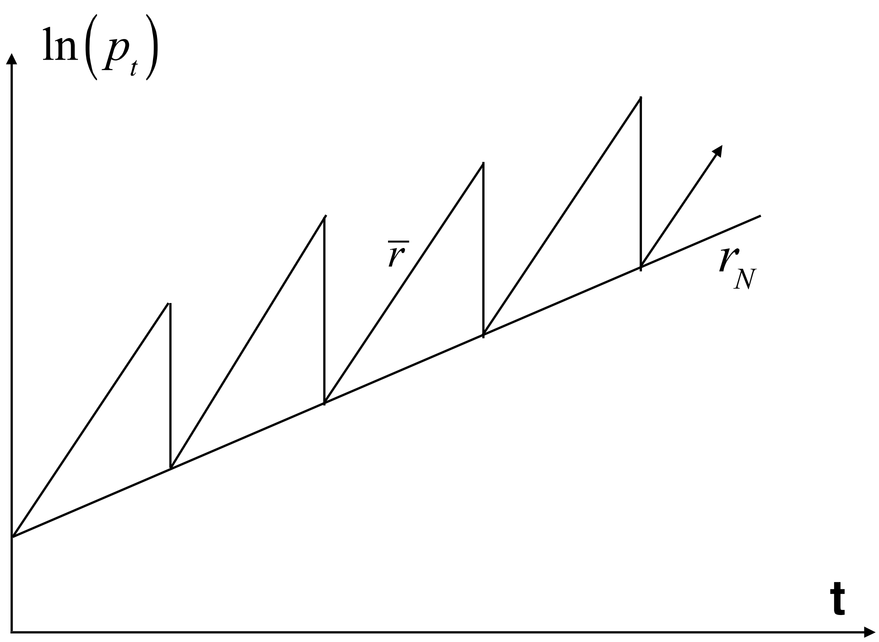


Figure B1: schematic representation of the price process, which grows instantaneously at the growth rate $\bar{r} > r_N$ and crashes periodically with an amplitude $k \ \ln \frac{p_t}{\exp(r_N t)}$ proportional with a coefficient $k \leq 1$ to the mispricing with respect to the average fundamental price process with return r_N. This "efficient crash" condition ensures that the long-term growth is given by r_N.

In terms of $X_t := \ln\left(p_t\right)$and $Y_t := \ln\left(p_t^N\right)$ and, changing the time scale in units of T, so that t= 0, T, 2T, 3T, … corresponds to t=0, 1, 2, 3, …, this model can be written as follows:

(B1) $$X_{t+1} = X_t + \bar{r}T - k\left(X_t - Y_t\right)$$

(B2) $$Y_{t+1} = Y_t + r_N T$$

This yields

(B3) $$X_{t+1} = (1-k)X_t + \bar{r}T + k(r_N T)t$$

(B4) $$Y_t = (r_N T)t$$

Expression (B4) is the trivial recovery that the log-price of the normal process increases by $r_N T$ over each period T.

For perfectly "efficient" crashes that bring back regularly the price at exactly the normal price at the instants nT, n=1, 2,…, the coefficient k is then exactly equal to 1. For k=1, expression (B3) simplifies to $X_t = \bar{r}T + (r_N T)\ t$, which grows with the normal price at the same growth rate r_N, up to a translation due to the transient from time 0 to T.

For k<1, the solution is less obvious and needs a more careful examination. The series (B3) can be solved using the formalism of generating probability function (GPF). Let us consider the general equation

(B5) $$X_t = aX_{t-1} + \alpha t + \beta\ .$$

We thus have a=1-k, $\beta = \bar{r}T$ and $\alpha = k(r_N T)$.

We introduce the GPF

(B6) $$P(z) = \sum_{t=0}^{\infty} X_t z^t\ ,$$

Multiplying (B5) by z^t and summing over t leads after some simple summations of series to the equation for P(z):

(B7) $$P(z) = X_0 + azP(z) + \frac{\alpha z}{(1-z)^2} + \frac{\beta z}{1-z}\ ,$$

whose solution is obviously

(B8) $$P(z) = \frac{X_0}{1-az} + \frac{\alpha z}{(1-az)(1-z)^2} + \frac{\beta z}{(1-az)(1-z)} \ ,$$

Expanding the r.h.s. of expression (B8) in series of integer powers of z and identifying term by term with the definition (B6) leads to the general solution:

(B9) $$X_t = X_0 a^t + \alpha \frac{t(1-a) - a + a^{t+1}}{(1-a)^2} + \beta \frac{1-a^t}{1-a} \quad .$$

Let us take *0<k<1*, then *0<a=1-k<1* so that a^t converges to 0 exponentially fast. Thus, at long times, (B9) reduces with an excellent approximation to

(B10) $$X_t = \alpha \frac{t(1-a) - a}{(1-a)^2} + \beta \frac{1}{1-a} \quad .$$

This solution (B10) can be checked by replacing directly in equation (B5). One can also use a simpler route than the full generating probability function formalism, which consists in searching for a solution of the form $X_t = mt + b$. Replacing in (B5) yields m=α/(1-a) and b=β/(1-a) – aα(1-a)^2, which recovers the exact (B10). One should note that this linear ansatz provides only the asymptotic shape of the solution, while the generating probability function formalism gives additionally the structure of the transient dynamics stemming from the initial condition.

The return of the price is asymptotically given by $\frac{1}{t} X_t$ which yields

(B11) $$\lim_{t\to\infty} \frac{1}{t} \ln\left(p_t\right) := \lim_{t\to\infty} \frac{1}{t} X_t = \alpha \frac{1}{1-a} \ .$$

With *a=1-k* and $\alpha = k(r_N T)$, this yields that the long-term average return is equal to $r_N T$ when time is counted discretely in units of T. Thus, notwithstanding the fact that the crash is only a fraction k<1 of the bubble size $\ln \frac{p_t}{\exp(r_N t)}$, the long-term average return of the periodically collapsing price is equal to the return r_N of the normal price. In other words, the price of the risky asset grows at the same long-term growth rate as the smooth normal price, even if it grows instantaneously as the faster rate $\bar{r} > r_N$.

Appendix C:

Proposition 3: $L(\lambda)$ is defined and there exits λ_t^A and λ_t^B such that it is a strictly concave function of $\lambda \in \Omega$ with $\Omega = \left\{ \lambda \middle| \lambda_t^A \le \lambda \le \lambda_t^B \text{ with } \lambda_t^L \le \lambda_t^A < 0 < \lambda_t^B \le \lambda_t^U \right\}$ provided $\sigma > 0$ or if $\rho \ne 0$ and either $\bar{r} - r_f + \frac{\sigma^2}{2} \ne 0$ or $\bar{\mathrm{K}} \ln(q_t) + r_D - r_f + \frac{\sigma^2}{2} \ne 0$.

Proof:

As in (14), we split $L(\lambda)$ (dropping the *t* subscript on λ), into two terms and consider the first term

$$L_1(\lambda) = 2\ln\left[e^{r_f}\left(1+\lambda\left(e^z-1\right)\right)\right] - \frac{1}{2}\ln\left(e^{2r_f}\left[\left(1+\lambda\left(e^z-1\right)\right)^2 + \lambda^2 e^{2z}\left(e^{\sigma^2}-1\right)\right]\right)$$

(C1) $$= r_f + 2\ln\left[1+\lambda\left(e^z-1\right)\right] - \frac{1}{2}\ln\left(\left(1+\lambda\left(e^z-1\right)\right)^2 + \lambda^2 e^{2z}\left(e^{\sigma^2}-1\right)\right)$$

with $z = \bar{r} - r_f + \frac{\sigma^2}{2}$

Then we have:

(C2) $$\frac{\partial L_1}{\partial \lambda} = \frac{2\left(e^z-1\right)}{1+\lambda\left(e^z-1\right)} - \frac{\left(1+\lambda\left(e^z-1\right)\right)\left(e^z-1\right) + \lambda e^{2z}\left(e^{\sigma^2}-1\right)}{\left(1+\lambda\left(e^z-1\right)\right)^2 + \lambda^2 e^{2z}\left(e^{\sigma^2}-1\right)}$$

And

(C3) $$\begin{aligned}\frac{\partial^2 L_1}{\partial \lambda^2} &= -2\frac{\left(e^z-1\right)^2}{\left(1+\lambda\left(e^z-1\right)\right)^2} \\ &\quad - \frac{\left[\left(1+\lambda\left(e^z-1\right)\right)^2 + \lambda^2 e^{2z}\left(e^{\sigma^2}-1\right)\right]\left[\left(e^z-1\right)^2 + e^{2z}\left(e^{\sigma^2}-1\right)\right]}{\left[\left(1+\lambda\left(e^z-1\right)\right)^2 + \lambda^2 e^{2z}\left(e^{\sigma^2}-1\right)\right]^2} \\ &\quad + \frac{\left[\left(1+\lambda\left(e^z-1\right)\right)\left(e^z-1\right) + \lambda e^{2z}\left(e^{\sigma^2}-1\right)\right]\left[2\left(1+\lambda\left(e^z-1\right)\right)\left(e^z-1\right) + 2\lambda e^{2z}\left(e^{\sigma^2}-1\right)\right]}{\left[\left(1+\lambda\left(e^z-1\right)\right)^2 + \lambda^2 e^{2z}\left(e^{\sigma^2}-1\right)\right]^2}\end{aligned}$$

Thusly

(C4)
$$\begin{aligned}
L_1(0) &= r_f \\
\left.\frac{\partial L_1}{\partial \lambda}\right|_{\lambda=0} &= e^z - 1 \\
\left.\frac{\partial^2 L_1}{\partial \lambda^2}\right|_{\lambda=0} &= -\left(e^z - 1\right)^2 - e^{2z}\left(e^{\sigma^2} - 1\right)
\end{aligned}$$

And we have for $L(\lambda)$ that

$$\begin{aligned}
L(0) &= r_f \\
\left.\frac{\partial L}{\partial \lambda}\right|_{\lambda=0} &= (1-\rho)e^z + \rho e^z - 1 \\
\left.\frac{\partial^2 L}{\partial \lambda^2}\right|_{\lambda=0} &= -(1-\rho)\left[\left(\exp\left(\overline{r}_t + \frac{\sigma^2}{2}\right) - 1\right)^2 + \exp\left(2\overline{r}_t + \sigma^2\right)\left(e^{\sigma^2} - 1\right)\right] \\
&\quad -\rho\left[\left(\exp\left(\kappa_i \ln(q_t) + r_D + \frac{\sigma^2}{2}\right) - 1\right)^2 + \exp\left(2\kappa_i \ln(q_t) + 2r_D + \sigma^2\right)\left(e^{\sigma^2} - 1\right)\right]
\end{aligned}$$

QED

Appendix D:

Proposition 4: We can approximate an optimal λ_t^* by

$$\lambda^* \approx \frac{\tilde{D} - 1 + \frac{\tilde{D}\sigma^2}{2}}{(1-\rho)\left(\tilde{A}-1\right)^2 + \rho\left(\tilde{B}-1\right)^2 + H_2 + H_3}$$

with

$$\begin{aligned}
\tilde{A} &\equiv \exp\left(\overline{r} - r_f\right) \\
\tilde{B} &\equiv \exp\left(\overline{K}\ln\left(q_t\right) + r_D - r_f\right) \\
\tilde{D} &\equiv (1-\rho)\tilde{A} + \rho\tilde{B} \\
H_2 &= \left(2\left((1-\rho)\tilde{A}^2 + \rho\tilde{B}^2\right) - \tilde{D}\right)\sigma^2 \\
H_3 &= \left((1-\rho)\tilde{A}^2 + \rho\tilde{B}^2\right)\frac{3\sigma^4}{4}
\end{aligned}$$

We can further approximate $\tilde{A}$, $\tilde{B}$, and $\tilde{D}$ by $\tilde{A} \approx 1 + \overline{r} - r_f$, $\tilde{B} = 1 + \overline{K}\ln\left(q_t\right) + r_D - r_f$, and $\tilde{D} \approx 1 + r_D - r_f$ so that we have using $2\left((1-\rho)\tilde{A}^2 + \rho\tilde{B}^2\right) - \tilde{D} \approx 1 + 3\left(\overline{r} - r_f + \overline{K}\ln\left(q_t\right) + r_D - r_f\right)$ that yields

$$\lambda^* \approx \frac{r_D - r_f + \frac{\sigma^2}{2}}{\sigma^2 + (1-\rho)\left(\overline{r} - r_f\right)^2 + \rho\left(\overline{K}\ln\left(q_t\right) + r_D - r_f\right)^2}$$

Proof:

From (9) and (10), we have:

$$L\left(\lambda_t\right) \equiv E\left[\ln\left(\frac{w_{t+1}}{w_t}\right)\right] = E\left[\ln\left(\exp\left(r_f\right)\left(1 + \lambda_t\left(\exp\left(\overline{a}_t + \sigma\varepsilon_t\right) - 1\right)\right)\right)\right]$$

(D1)
$$= \frac{1-\rho}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[\ln\left(\exp\left(r_f\right)\left[1 + \lambda_t\left(\tilde{A}\exp(\sigma\varepsilon) - 1\right)\right]\right)\right]\exp\left(-\frac{\varepsilon^2}{2}\right)d\varepsilon$$
$$+ \frac{\rho}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[\ln\left(\exp\left(r_f\right)\left[1 + \lambda_t\left(\tilde{B}\exp(\sigma\varepsilon) - 1\right)\right]\right)\right]\exp\left(-\frac{\varepsilon^2}{2}\right)d\varepsilon$$

And by the RE condition:

(D2) $$\tilde{D} \approx 1 + r_D - r_f$$

As $\bar{r}_t$ can be large, we use the following second order expansions for log and exponential:

(D3) $$\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 \qquad |x-1| \le 1, \;\; x \ne 0$$
$$\exp(x) \approx 1 + x + \frac{x^2}{2}$$

Because of (13), the log arguments are bounded away from zero, and then the expressions (12) becomes:

(D4) $$E\left[\ln\left(\frac{w_{t+1}}{w_t}\right)\right] \approx$$
$$\frac{1-\rho}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\ln\left(\lambda_t\left(\tilde{A}\left(1+\sigma\varepsilon+\frac{\sigma^2\varepsilon^2}{2}\right)-1\right)+1\right)\right]\exp\left(-\frac{\varepsilon^2}{2}\right)d\varepsilon$$
$$+\frac{\rho}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\ln\left(\lambda_t\left(\tilde{B}\left(1+\sigma\varepsilon+\frac{\sigma^2\varepsilon^2}{2}\right)-1\right)+1\right)\right]\exp\left(-\frac{\varepsilon^2}{2}\right)d\varepsilon$$
$$+r_f$$

We expand the integrand using:

(D5) $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varepsilon^m \exp\left(-\frac{\varepsilon^2}{2}\right)d\varepsilon = \begin{cases} 0 & m \text{ odd} \\ 2^{-m/2}\dfrac{m!}{(m/2)!} & m \text{ even} \end{cases}$$

which gives for the first integrand:

(D6) $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\ln\left(\lambda_t\left(\tilde{A}\left(1+\sigma\varepsilon+\frac{\sigma^2\varepsilon^2}{2}\right)-1\right)+1\right)\right]\exp\left(-\frac{\varepsilon^2}{2}\right)d\varepsilon$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left[\lambda_t\left(\tilde{A}\left(1+\sigma\varepsilon+\frac{\sigma^2\varepsilon^2}{2}\right)-1\right) - \frac{1}{2}\lambda_t^2\left(\tilde{A}\left(1+\sigma\varepsilon+\frac{\sigma^2\varepsilon^2}{2}\right)-1\right)^2\right]\exp\left(-\frac{\varepsilon^2}{2}\right)d\varepsilon$$
$$\approx$$

Expanding the first integrand and dropping the subscript *t* gives:

(D7)

$$
\begin{aligned}
&\ln\left(\lambda\left(\tilde{A}\left(1+\sigma\varepsilon+\frac{\sigma^2\varepsilon^2}{2}\right)-1\right)+1\right)\\
&\approx \lambda\left(\tilde{A}\left(1+\sigma\varepsilon+\frac{\sigma^2\varepsilon^2}{2}\right)-1\right)-\frac{1}{2}\lambda^2\left(\tilde{A}\left(1+\sigma\varepsilon+\frac{\sigma^2\varepsilon^2}{2}\right)-1\right)^2\\
&=\lambda\left(\tilde{A}-1\right)-\frac{1}{2}\lambda^2\left(1-2\tilde{A}+\tilde{A}^2\right)\\
&+\varepsilon\left(\lambda\tilde{A}\sigma-\frac{1}{2}\lambda^2\left(2\sigma\tilde{A}^2-2\tilde{A}\sigma\right)\right)\\
&+\varepsilon^2\left(\lambda\tilde{A}\frac{\sigma^2}{2}-\frac{1}{2}\lambda^2\left(-\tilde{A}\sigma^2+2\sigma^2\tilde{A}^2\right)\right)\\
&+\varepsilon^3\left(-\frac{1}{2}\lambda^2\left(\sigma^3\tilde{A}^2\right)\right)\\
&+\varepsilon^4\left(-\frac{1}{2}\lambda^2\left(\frac{\sigma^4\tilde{A}^2}{4}\right)\right)
\end{aligned}
$$

Therefore, the first integrand is given by:

(D8)

$$
\begin{aligned}
&\lambda\left(\tilde{A}-1+\tilde{A}\frac{\sigma^2}{2}\right)\\
&-\frac{1}{2}\lambda^2\left(1-2\tilde{A}+\tilde{A}^2-\tilde{A}\sigma^2+2\sigma^2\tilde{A}^2+\frac{3\sigma^4\tilde{A}^2}{8}\right)
\end{aligned}
$$

The second integrand is of the same form so that we have:

(D9)

$$
\begin{aligned}
&\frac{\partial L(\lambda)}{\partial \lambda} \approx \\
&\approx (1-\rho)\left[\tilde{A}-1+\frac{\tilde{A}\sigma^2}{2}-\lambda\left(\sigma^2\left(2\tilde{A}^2-\tilde{A}\right)+\left(\tilde{A}-1\right)^2+\frac{3\tilde{A}^2\sigma^4}{4}\right)\right] \\
&+\rho\left[\tilde{B}-1+\frac{\tilde{B}\sigma^2}{2}-\lambda\left(\sigma^2\left(2\tilde{B}^2-\tilde{B}\right)+\left(\tilde{B}-1\right)^2+\frac{3\tilde{B}^2\sigma^4}{4}\right)\right] \\
&=\tilde{D}\left(1+\frac{\sigma^2}{2}\right)-1-\lambda\begin{pmatrix}(1-\rho)\left(\tilde{A}-1\right)^2+\rho\left(\tilde{B}-1\right)^2 \\ +\left(\left((1-\rho)2\tilde{A}^2+2\rho\tilde{B}^2\right)-\tilde{D}\right)\sigma^2 \\ +\left((1-\rho)\tilde{A}^2+\rho\tilde{B}^2\right)\frac{3\sigma^4}{4}\end{pmatrix}
\end{aligned}
$$

Note that $\tilde{D}\left(1+\frac{\sigma^2}{2}\right)-1 \approx r_D - r_f + \frac{\sigma^2}{2}$

The approximation is a strictly concave function if $(1-\rho)\left(\tilde{A}-1\right)^2+\rho\left(\tilde{B}-1\right)^2+H_2+H_3>0$. Assuming this to be the case, we set $\frac{\partial \bar{E}(\lambda)}{\partial \lambda}=0$ and calculate:

(D10)

$$
\lambda^* = \frac{\tilde{D}\left(1+\frac{\sigma^2}{2}\right)-1}{(1-\rho)\left(\tilde{A}-1\right)^2+\rho\left(\tilde{B}-1\right)^2+H_2+H_3}
$$

QED

References

Abreu, Dilip and Markus K. Brunnermeier. 2002. "Synchronization risk and delayed arbitrage", Journal of Financial Economics, 66 (2002) 341–360.

Akerlof, George A. and Robert J. Shiller, Animal spirits (how human psychology drives the economy, and why it matters for global capitalism), Princeton University Press, Princeton and Oxford, 2009.

Ait-Sahalia, Yacine and Jean Jacod. 2012. "Analysing the Spectrum of Asset Returns: Jump and Volatility Components in High Frequency data", Journal of Economic Literature, 50(4), 1007-1050.

Ait-Sahalia, Yacine and Jean Jacod. 2009. "Testing for Jumps in a Discretely Observed Process", The Annals of Statistics, 37 (1), 184-222.

Ait-Sahalia. 2004. "Disentangling diffusion from jumps", Journal of Financial Economics 74, 487-528.

Ait-Sahalia, Yacine and Julio Cacho-Diaz, and T. R. Hurd. Portfolio choice with jumps: a closed-form solution. The Annals of Applied Probability, 19(2), 556-584, 2009.

Ambarish, Ramasastry and Lester Seigel. 1996. "Time is the Essence", Risk 9 (8), August.

Andersen, Torben G., Tim Bollerslev, Francis X. Diebold, 2007. Roughing it up: Including jump components in the measurement, modeling, and forecasting of return volatility, The review of economics and statistics 89 (4), 701-720.

Andruszkiewicz, G, M. Davis And S. Lleo. 2017. "Risk-Sensitive Investment in a finite-factor model", Stochastics 89, 89-114.

Ardila-Alvarez, Diego, Zalan Forro and Didier Sornette, The Acceleration effect and Gamma factor in Asset Pricing, Swiss Finance Institute Research Paper No. 15-30. Available at SSRN: http://ssrn.com/abstract=2645882, 2016.

Audrino, Francesco and Yujia Hu. 2016. "Volatility Forecasting: Downside Risk, Jumps and Leverage Effect", Econometrics, 4, 8, doi:10.3390/econometrics4010008, pp. 1-24

Bali, Turan G. and Hao Zhou. 2014. "Risk, uncertainty, and Expected Returns".

Bao, Te; Cars Hommes and Tomasz Makarewicz. 2014. "Bubble Formation and (In)Efficient Markets in Learning-to-Forecast and – Optimize Experiments", University of Groningen, The Netherlands.

Barndorff-Nielsen; Svend Erik Graversen; Jean Jacod; and Neil Shephard. 2005. "Limit theorems for bipower variation in financial econometrics".

Black, Fischer, 1986. Noise, Journal of Finance 41 (3), 20-30.

Blanchard, Olivier J., 1979, Speculative bubbles, crashes, and rational expectations, Economics Letters 3, 387–389.

Blanchard, Olivier J., and Mark W. Watson, 1982. "Bubbles, Rational Expectations and Speculative Markets", (In Crisis in Economic and Financial Structure: Bubbles, Bursts, and Shocks, edited by Paul Wachtel. Lexington: Lexington Books).

Breiman, L. 1961. "Optimal Gambling Systems for Favorable Games", *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability*, 1, 63-68.

Brunnermeier, Markus K. 2008. "Bubbles", in the *New Palgrave Dictionary of Economics*, Second Edition.

Brunnermeier, M.K. and M. Oehmke, 2013. Bubbles, financial crises, and systemic risk. Handbook of the Economics of Finance, Chapter 18, Elsevier B.V., 1221-1288.

Corsi, Fulvio and Didier Sornette, 2014. Follow the money: The monetary roots of bubbles and crashes, International Review of Financial Analysis 32, 47-59.

Camerer, C., Bubbles and fads in asset prices. Journal of Economic Surveys, 1989, 3(1):3–41.

Davis, M and S. Lleo. 2013a. "Fractional Kelly Strategies in Continuous Time: Recent Developments." Handbook of the Fundamentals of Financial Decision Making. Leonard C. MacLean and William. T. Ziemba editors. World Scientific Publishing Co.

Davis, M and S. Lleo. 2015a. Jump-Diffusion Asset-Liability Management Via Risk-Sensitive Control. OR Spectrum 37, 655. doi:10.1007/s00291-014-0371-x.

Davis, Mark H. A. and Sebastien Lleo. 2015b. Risk-Sensitive Investment Management, World Scientific Publishing Co. Pte. Ltd., Singapore.

Demos, G. and D. Sornette. 2016. "Birth or burst of financial bubbles: which one is easier to diagnose?" Quantitative Finance, 17:5, 657-675.

Elton, Edwin J. and Martin J. Gruber. 1974. "On the Maximization of the Geometric Mean with Lognormal Return Distribution", Management Science 21 (4), 483-488.

Evanoff, Douglas D.; George G. Kaufman; and A. G. Malliaris editors. 2012. New Perspectives on Asset Price Bubbles: Theory, Evidence, and Policy, Oxford University Press, New York.

Evans, L. L. jr. 2003. Why the bubble burst – US stock market performance since 1982. Edward Elgar Publishing Limited.

Evans, G. W., 1991, Pitfalls in testing for explosive bubbles in asset prices, American Economic Review 81, 922–930.

Fama, Eugene F. 1988. Perspectives on October 1987, or, What Did We Learn from the Crash? Center for Research in Security Prices Working paper series no. 232.; Chicago: Graduate School of Business University of Chicago.

Gürkaynak, Refet S. 2008. "Econometric Tests of Asset Price Bubbles: Taking Stock", Journal of Economic Surveys 22 (1), 166-186.

Harras, G. and D. Sornette 2011. How to grow a bubble: A model of myopic adapting agents, Journal of Economic Behavior and Organization 80 (1), 137-152.

Homm, Ulrich and Jörg Breitung, 2012. Testing for speculative bubbles in stock markets: A comparison of alternative methods. Journal of Financial Econometrics 10 (1), 198-231.

Huang, Xin and George Tauchen. 2005. "The Relative Contribution of Jumps to Total Price Variance". Journal of Financial Econometrics 3 (4), 456-499.

Hüsler, Andreas, Didier Sornette and Cars H. Hommes, 2013. Super-exponential bubbles in lab experiments: evidence for anchoring over-optimistic expectations on price, Journal Economic Behavior and Organization 92, 304-316.

Jacquier, Eric and Cedric Okou. 2014. "Disentangling Continuous Volatility from Jumps in Long-Run Risk-Return Relationships", Journal of Financial Econometrics 12 (3), 544-583.

Jarrow, Robert. 2016a. "Testing for Asset Price Bubbles: Three New Approaches", Quantitative Finance Letters 4 (1), 4-9, DOI: 10.1080/21649502.2015.1165838, 2016.

Jarrow, Robert. 2016b. "Bubbles and Multiple-Factor Asset Pricing Models", International Journal of Theoretical and Applied Finance 19 (1), 1650007 [19 pages].

Jiang, Zhi-Qiang, Wei-Xing Zhou, Didier Sornette, Ryan Woodard, Ken Bastiaensen, Peter Cauwels, 2010. Bubble Diagnosis and Prediction of the 2005-2007 and 2008-2009 Chinese stock market bubbles, Journal of Economic Behavior and Organization 74, 149-162.

Johansen, Anders, Olivier Ledoit and Didier Sornette, 2000. Crashes as critical points, International Journal of Theoretical and Applied Finance 3 (2), 219-255.

Johansen, Anders, and Didier Sornette, 2001/2002, Large stock market price drawdowns are outliers, Journal of Risk 4, 69-110.

Johansen, A. and D. Sornette, 2010. Shocks, crashes and bubbles in financial markets. Brussels Economic Review (Cahiers economiques de Bruxelles) 53 (2), 201-253.

Johansen, Anders, Didier Sornette and Olivier Ledoit, 1999. Predicting Financial Crashes using discrete scale invariance, Journal of Risk 1 (4), 5-32.

Kaizoji, T. and D. Sornette. "Market Bubbles and Crashes", in the Encyclopedia of Quantitative Finance (Wiley, 2010), http://www.wiley.com//legacy/wileychi/eqf/ (long version at http://arXiv.org/abs/0812.2449)

Kelly, J. R. 1956. "A New Interpretation of Information Rate", *Bell Systems Technical Journal*, 35, 917-926.

Kindleberger, Charles P. 2005. Manias, Panics, and Crashes: A History of Financial Crises (Wiley; 5 edition).

Kreuser, Jerome L. 2014. "Risk Management in Public Institutions is Different: Everyone is Different in its Own Way", March 11, ETH Zurich Risk Center Seminar, http://www.RisKontroller.com .

Leiss, Matthias, Heinrich H. Nax and Didier Sornette, 2015. Super-Exponential Growth Expectations and the Global Financial Crisis, Journal of Economic Dynamics and Control 55, 1-13.

Levy, H. and H. M. Markowitz, 1979. "Approximating Expected Utility by a Function of Mean and Variance", The American Economic Review 69 (3), 308-317.

Lin, L. And D. Sornette. 2013. "Diagnostics of Rational Expectation Financial Bubbles with Stochastic Mean-Reverting Termination Times", The European Journal of Finance 19 (5-6) 344-365.

Lin, Li.; Ruo En Ren; and Didier Sornette. 2014. The Volatility-Confined LPPL Model: A Consistent Model of "Explosive" Financial Bubbles with Mean-Reversing Residuals, International Review of Financial Analysis 33, 210-225.

Lux, Thomas. 1995. "Herd Behaviour, Bubbles and Crashes", The Economic Journal 105 (431), 881-896.

Lux, Thomas, and Didier Sornette. 2002. "On rational bubbles and fat tails", Journal of Money, Credit and Banking Part 1 34, 589–610.

Malevergne Yannick and Didier Sornette, 2001. Multi-dimensional Rational Bubbles and fat tails, Quantitative Finance 1, 533-54.

MacLean, L. C.; Edward O. Thorp; and W.T. Ziemba. 2010a. *The Kelly Capital Growth, Investment Criterion: Theory and Practice*, World Scientific Handbook in Financial Economic Series, World Scientific Publishing C. Pte. Ltd., Singapore.

MacLean, L.C. and William T. Ziemba. 2010. "Using the Kelly criterion for investing", *Stochastic Optimization Methods in Finance and Energy,* Volume 163 of the series International Series in Operations Research & Management Science pp 3-20, Springer, 25 July 2011.

Meltzer, Allan H. 2002. "Rational and Irrational Bubbles", Keynote address for the Federal Reserve Bank of Chicago – World Bank Conference on Asset Price Bubbles, Chicago, April 23.

Merton, Robert C. 1971. "Optimal consumption and portfolio rules in a continuous time model", Journal of Economic Theory, 3:373-4113.

Merton, Robert C. 1980. "On Estimating the Expected Return on the Market", Journal of Financial Economics 8, 323-361.

Merton, Robert C. 1990. Continuous-Time Finance, Blackwell Publishers Inc., Cambridge, Massachusetts.

Phillips, Peter C.B. and Jun Yu. 2011. "Dating the Timeline of Financial Bubbles During the Subprime Crisis", Quantitative Economics 2, (3), 455-491.

Phillips, Peter C. B.; Yangru Wu; and Jun Yu. 2009. "Explosive Behaviour in the 1990s Nasdaq: When Did Exuberance Escalate Asset Values?", Cowles Foundation Discussion Paper No. 1699, http://ssrn.com/abstract=1413830.

Phillips, Peter C. B., Shuping Shi, and Jun Yu, 2015. Testing for multiple bubbles: historical episodes of exuberance and collapse in the S&P500. International Economic Review 56 (4), 1043-1078.

Platen, Eckhard and Nicola Bruti-Liberati. 2010. Numerical Solution of Stochastic Differential Equations with Jumps in Finance, Springer-Verlag, Heidelberg.

Ren, Yu; Cong Xiong and Yufei Yuan. 2012. "House Price Bubbles in China", The Wang Yanan Institute for Studies in Economics, Xiamen University, Fujian, China, March 27.

Scheinkman, Jose A. 2014. Speculation, Trading, and Bubbles, Kenneth J. Arrow Lecture Series, Columbia University Press, New York.

Scherbina, Anna. 2013. "Asset Price Bubbles: Selective Survey," IMF Working Paper, WP/13/45, IMF Washington, DC.

Sornette, D. 2002. "Slimming" of power law tails by increasing market returns, Physica A 307 (1-2), 63-106.

Sornette, D., 2003, *Why Stock Markets Crash: Critical Events in Complex Financial Systems* (Princeton, NJ: Princeton University Press).

Sornette, D, 2003. Critical market crashes. Physics Reports 378 (1), 1-98.

Sornette, D. 2009. "Dragon-Kings, Black Swans and the Prediction of Crises", International Journal of Terraspace Science and Engineering 2 (1), 1-18 (http://ssrn.com/abstract=1470006)

Sornette. Physics and financial economics (1776-2014): Puzzles, ising and agent-based models. Rep. Prog. Phys. 77,62001, 2014.

Sornette, Didier and J.V. Andersen. 2002. "A Nonlinear Super-Exponential Rational Model of Speculative Financial Bubbles", Int. J. Mod. Phys. C 13 (2), 171-188.

Sornette, Didier and Rama Cont. 1997. Convergent Multiplicative Processes Repelled from Zero: Power Laws and Truncated Power Laws, J. Phys. I France 7, 431-444.

Sornette, D. and G. Ouillon. 2012. "Dragon-kings: mechanisms, statistical methods and empirical evidence", Eur. Phys. J. Special Topics 205, 1-26 (special issue on power laws and dragon-kings) (http://ssrn.com/abstract=2191590).

Sornette, D. and P. Cauwels. 2015. "Financial bubbles: mechanisms and diagnostics", Review of Behavioral Economics 2 (3), 279-305.

Sornette, Didier, Christian Vanneste and Anne Sornette, 1991. Dispersion of b-values in Gutenberg-Richter law as a consequence of a proposed fractal nature of continental faulting, Geophys. Res. Lett. 18, 897-900.

Sornette, Didier, Ryan Woodard, Maxim Fedorovsky, Stefan Riemann, Hilary Woodard, Wei-Xing Zhou (The Financial Crisis Observatory), 2009. The Financial Bubble Experiment: advanced diagnostics and forecasts of bubble terminations (http://arxiv.org/abs/0911.0454).

Sornette, Didier, Ryan Woodard, Maxim Fedorovsky, Stefan Reimann, Hilary Woodard, Wei-Xing Zhou, 2010. (The Financial Crisis Observatory), The Financial Bubble Experiment: Advanced Diagnostics and Forecasts of Bubble Terminations Volume II--Master Document (beginning of the experiment) (http://arxiv.org/abs/1005.5675).

Sornette, Didier, Ryan Woodard, Wanfeng Yan and Wei-Xing Zhou, 2013a. Clarifications to Questions and Criticisms on the Johansen-Ledoit-Sornette bubble Model, Physica A 392 (19), 4417-4428.

Sornette Didier and Wei-Xing Zhou, 2006. Predictability of Large Future Changes in major financial indices, International Journal of Forecasting 22, 153-168.

Taipalus, Katja. 2012. "Detecting asset price bubbles with time-series methods", Scientific monograph E:47, Bank of Finland, Helsinki.

Tauchen, George and Hao Zhou. 2011. "Realized jumps on financial markets and predicting credit spreads", Journal of Econometrics, 160, 102-118.

Thorp, Edward O. (2006). The Kelly criterion in blackjack, sports betting and the stock market. In S. A. Zenios and W. T. Ziemba (Eds.), Handbook of asset and liability management, Handbooks in Finance, pp. 385–428. North Holland.

Thorp. Edward O. 2010. "Portfolio Choice and the Kelly Criterion", Chapter 7 in MacLean et. al., 2010. Xiong, James X. and Roger G. Ibbotson. 2013. "Momentum, Acceleration, and Reversal, Ibbotson Associates, Chicago.

Vogel, Harold L. and Richard A. Werner, 2015. An analytical review of volatility metrics for bubbles and crashes. International Review of Financial Analysis 38, 15-28.

Woodard, Ryan, Didier Sornette, Maxim Fedorovsky, 2010. The Financial Bubble Experiment: Advanced Diagnostics and Forecasts of Bubble Terminations, Volume III (beginning of experiment + post-mortem analysis) (http://arxiv.org/abs/1011.2882).

Xiong, W., 2013. Bubbles, crises, and heterogeneous beliefs. Chapter 24, Handbook on Systemic Risk, Edited by Jean-Pierre Fouque, Joseph A. Langsam, Cambridge University Press, 663-713.

Yan, Wanfeng; Reda Rebib; Ryan Woodard; Didier Sornette. 2012. "Detection of Crashes and Rebounds in Major Equity Markets", International Journal of Portfolio Analysis & Management (IJPAM) 1(1), 59-79. io Analysis & Management (IJPAM) 1(1), 59-79.

Yan, Wanfeng; Ryan Woodard and Didier Sornette. 2010a. "Diagnosis and Prediction of Tipping Points in Financial Markets: Crashes and Rebounds", Physics Procedia 3 (5), 1641-1657 (http://arxiv.org/abs/1001.0265).

Yan, Wanfeng; Ryan Woodard and Didier Sornette. 2010b. "Inferring Fundamental Value and Crash Nonlinearity from Bubble Calibration", Quantitative Finance 14 (7), 1273-1282, 2014.

Young, Terry W. 1991. Calmar ratio: a smoother tool, Futures, 20 (1), 40.

Ziemba, William T. 2003. The Stochastic Programming Approach to Asset, Liability, and Wealth Management, The Research Foundation of AIMR.