Optimal Insurance and Reinsurance Portfolios, Implied Pricing, Allocating Retrocessional Costs and Capital Allocation[1]

Jerome Kreuser[2]
and
Morton Lane[3]

Abstract

Some reinsurers use optimization procedures to generate underwriting portfolios, maximizing expected returns which are perfectly aligned with their stated risk preferences. Similar objectives apply to those who use simulation or DFA techniques. However, beyond the optimal portfolio itself, optimizers as part of their output also generate marginal economic signals, such as "implied" or "risk adjusted" probabilities which are important but underused and often misunderstood management tools. The purpose of this paper is to further illustrate the power of those economic signals.

In an earlier paper[4] we illustrated how implied or risk-adjusted probabilities from optimal solutions may be derived and used in a simple single risk zone example. In this paper we continue the same simplifying universe but with multiple risk zones. We then use the marginal outputs to illustrate how to price indifference points for traditional retrocession purchases, which complement the optimum portfolio. In addition, we show how the implied probabilities may be used to allocate retrocession costs to the respective zones. Of course, allocating retrocession costs is an important sub-species of allocating capital costs in general. Actually we also believe the marginal outputs are the key to unlocking general capital allocation decisions.

An important part of that optimization and allocation process is the articulation of risk preferences of the optimizer. Implied or risk adjusted probabilities contain, the DNA, so to speak, of those risk preferences. That is why they are the appropriate metrics to use in allocation decisions. Importantly, in this paper we also begin some empirical exercises to reveal the risk preferences of reinsurers. In particular, we examine how the risk preferences of multi-line reinsurance companies are delegated to their cat reinsurance line and how that compares with the risk preferences of mono-line cat reinsurers.

---

[1] We are grateful for the support of Renaissance Re and AIR Worldwide in the furtherance of this effort. In addition much of the empirical work was with the support of Goldman Sachs Asset Management.
[2] Executive Director of The RisKontrol Group, Bern, Switzerland. kreuser@riskontroller.com
[3] President of Lane Financial LLC, Chicago, Illinois, USA: mlane@lanefinancialllc.com
[4] An Introduction to the Benefits of Optimization Models for Underwriting Portfolio Selection, Jerome Kreuser and Morton Lane, Proceedings of the 28th International Congress of Actuaries, Paris, June 2006.

Introduction

In our previous paper on extracting the power and value of implied probability outputs from optimizing[5] solutions to selecting underwriting portfolios, we provided an extensive numerical illustration using a single risk zone, Florida wind. The general mathematical model was given, together with its dual, to provide a basis for our computations, however we felt that numerical illustration of concepts was necessary to promote more general understanding. In this paper we continue the practice of numerical illustration[6], albeit expanded to three risk zones, with the purpose of disentangling the separate effects of those zones. In the process we hope to show how implied probabilities can be used to help allocate retrocessional costs to other zones.

To the extent that retrocession costs are allocated within a company, who should pick them up? If the retrocession is purchased to cover a whole book of business it is a cost and a benefit to each division. But who should pay that cost, if it is to be allocated to the divisions, and how much will that affect the profitability of the various divisions? Some divisions will utilize the potential recoveries more than others. Some expect to make more recoveries than others. So a typical corporate response is to allocate the cost according to expected recoveries. However, as we will illustrate, that allocation takes no account of the fact that certain outcomes are less desirable to management than others. It contains no allowance for management's risk preferences. Implied probabilities, on the other hand, specifically contain information about risk preference. Using them in the allocation process is more appropriate. As we will illustrate allocation should be by risk adjusted expected losses, not by simple expected loss. That way the division causing the least desirable outcomes should be the one picking up the major part of the cost.

Then again when allocations are made on this basis what does that say about the profitability of each division. It will certainly be different from allocation done on a simple expected loss basis.

Now allocating retrocessional cost is not a million miles away from the question of allocating capital to divisions. After all, retrocession is a form of capital. The allocation of retrocessional costs, therefore, gives insight into the allocation of capital as a whole.

We begin by repeating the structure of the general model before diverting to the numerical illustration.

The Problem

[5] The optimization model utilized herein is a proprietary model system – RisKontroller ReAL™ (RisKontroller Reinsurance Asset Liability Optimizer). The approach embedded in RisKontroller ReAL, is Dynamic Stochastic Programming (DSP), which can be used in a variety of applications. Other examples of Dynamic Stochastic Programming applications can be found in Ziemba and Mulvey (1998).

[6] The numerical data used in this illustration is not current. It was representative prior to the model revisions post Katrina, and was provided by AIR Worldwide Ltd. The post Katrina data, either long term or short term is now much different from the above.

The basic problem is to maximize expected return by selecting among a set of deal alternatives. The environment for the selection is a risky one in which each deal can have multiple outcomes, hence the selected portfolio can have a variety of outcomes. Some of these outcomes are less desirable than others. Management expresses its risk preference by stating certain restrictions for the potential portfolio outcomes. Typically these are Value at Risk (a.k.a. VaR) constraints, or Conditional Value at Risk[7] (a.k.a. CVaR) constraints. In addition, limits are placed on the practical deal sizes, whether assumed or ceded. The model inputs and decision variables are listed below.

Model Inputs	
Name	**Identification**
$\overline{bda}$	Bid/ask spread.
$\overline{c}_k$	Confidence level expressed as a decimal for risk level *k*
$\overline{capital}$	Starting capital
$\overline{cvar}_k$	Percent limit on loss of capital for risk level *k*
$\overline{loss}_{i,j}$	Unit loss of deal *j* in scenario *i*
$\overline{price}_j$	Price of deal *j* as a percent
$\overline{\rho}_i$	Probability of scenario *i*
$\overline{rate}$	Rate of return on investments
$\overline{tm}$	Percentage of capital as limit on total ceded premiums
$\overline{trs}$	Transaction costs as a percent

Decision Variables Determined by Model	
Name	**Identification**
α_k	Alpha value for risk *k*, which turns out to be VaR for active constraints
$deal_j$	Amount of premium of deal *j* written
$funds0$	Beginning period funds net of capital
$funds1_i$	End-of-period funds net of capital in scenario *i*
$gains_i$	Gains from recoveries in scenario *i*
$losses_i$	Losses in scenario *i*
$retro_j$	Amount of premium of deal *j* ceded
$z_{k,i}$	Excess loss over VaR of funds in scenario *i* for risk level *k*

Algebraically the optimizing problem can be expressed as:

[7] Conditional Value at Risk is also known as Tail Value at Risk (TVaR).

(1) $$\max_{\substack{funds0,\, funds1_i,\, z_i,\, \alpha_k,\\ deal_j,\, retro_j,\, losses_i,\, gains_i}} \sum_{i \in I} \overline{\rho}_i \, funds1_i$$ expected value of funds at end of period

Subject to;

(2) $$funds0 - \sum_{j \in J} \left((1-\overline{trs})(1-\overline{bda})\, price_j deal_j - (1+\overline{trs})(1+\overline{bda})\, price_j retro_j \right) = 0$$

Initial funds

(3) $$funds1_i - \left(1+\overline{rate}\right) funds0 + losses_i - gains_i = \overline{rate} \times \overline{capital}$$ End-of-period funds in scenario *i*

(4) $$losses_i - \sum_{j \in J} deal_j \overline{loss_{i,j}} = 0$$ Losses in scenario i

$$gains_i - \sum_{j \in J} retro_j \overline{loss_{i,j}} = 0$$ Gains in scenario i

Equations (2), (3), and (4) can be collapsed into one but we keep them separate here for ease of exposition.

(5) $$\sum_{j \in J} retro_j \le \overline{tm} \times \overline{capital}$$ Limit on retrocessions as a % of capital

(6) $$-funds1_i - \alpha_k - z_{k,i} \le 0$$ Value of excess loss by scenario i

$$\sum_{i \in I} \overline{\rho}_i z_{k,i} + \left(1-\overline{c}_k\right)\alpha_k \le \left(1-\overline{c}_k\right) \overline{cvar_k} \times \overline{capital}$$ k CVaR constraints

(7) Bounds on deals and non-negativity constraints

$0 \le deal_j \le \overline{\text{deal limit}_j}$ Limit on deal

$0 \le retro_j \le \overline{\text{retro limit}_j}$ Limit on retrocession

$-z_{k,i} \le 0$ Non-negativity constraint on excess loss

$funds0,\, funds1_i,\, \alpha_k$ can otherwise take any value.

We now begin identification of a specific numerical example to illustrate the optimizing features of our general model.

The Deal Opportunities

We assume a world of three wind zones: Florida, Georgia to Maine, and the Rest of the US. These we label FLA, GEO and RUS. In addition we allow the possibility of covers for Nation Wide Wind (NWW) which allows covers to be obtained covering the whole country, i.e., covering wind in FLA, GEO or RUS. Each of the deals is assumed to be of Industry Loss Warranty (ILW), or binary, form and each can be assumed or ceded although not to the same degree. The Table below contains the details:

Prices for given Deal Opportunities

INPUT PRICES	**Reinsurance Asset/Liability Optimizer Prices**			
Name	Florida Wind	Georgia to Maine Wind	Rest US Wind	Nationwide Wind
Tag	FlaWnd	GtMWnd	OthWnd	NWnd
Linkages				NWnd
Type	ILW	ILW	ILW	ILW
Distribution	FlaWnd	GtMWnd	OthWnd	FlaWnd,GtMWnd,OthWnd
Region	US	US	US	US
Trigger Point				
Billions	Wind	Wind	Wind	Wind
$1.0				
$1.5				
$2.0				
$3.0				
$5.0	26.00%	8.00%		28.00%
$7.5				
$10.0	17.50%	4.50%		20.00%
$12.5	15.00%	4.25%		17.00%
$15.0	13.00%	3.75%		14.50%
$20.0	9.50%	2.75%	2.00%	11.00%
$25.0	7.25%	2.00%		8.75%
$30.0	5.75%	1.75%		7.75%
$40.0	4.50%	1.50%		5.50%
$50.0	4.00%	1.00%		4.50%

Maximum written Limits

Name	Florida Wind	Georgia to Maine Wind	Rest US Wind	Nationwide Wind
Tag	FlaWnd	GtMWnd	OthWnd	NWnd
Linkages				NWnd
Type	ILW	ILW	ILW	ILW
Distribution	FlaWnd	GtMWnd	OthWnd	FlaWnd,GtMWnd,OthWnd
Region	US	US	US	US
Trigger Point				
Billions	Wind	Wind	Wind	Wind
$1.0				
$1.5				
$2.0				
$3.0				
$5.0	25	25		25
$7.5				
$10.0	25	25		25
$12.5	25	25		25
$15.0	25	25		25
$20.0	25	25	25	25
$25.0	25	25		25
$30.0	25	25		25
$40.0	25	25		25
$50.0	25	25		25

Maximum retrocession Limits

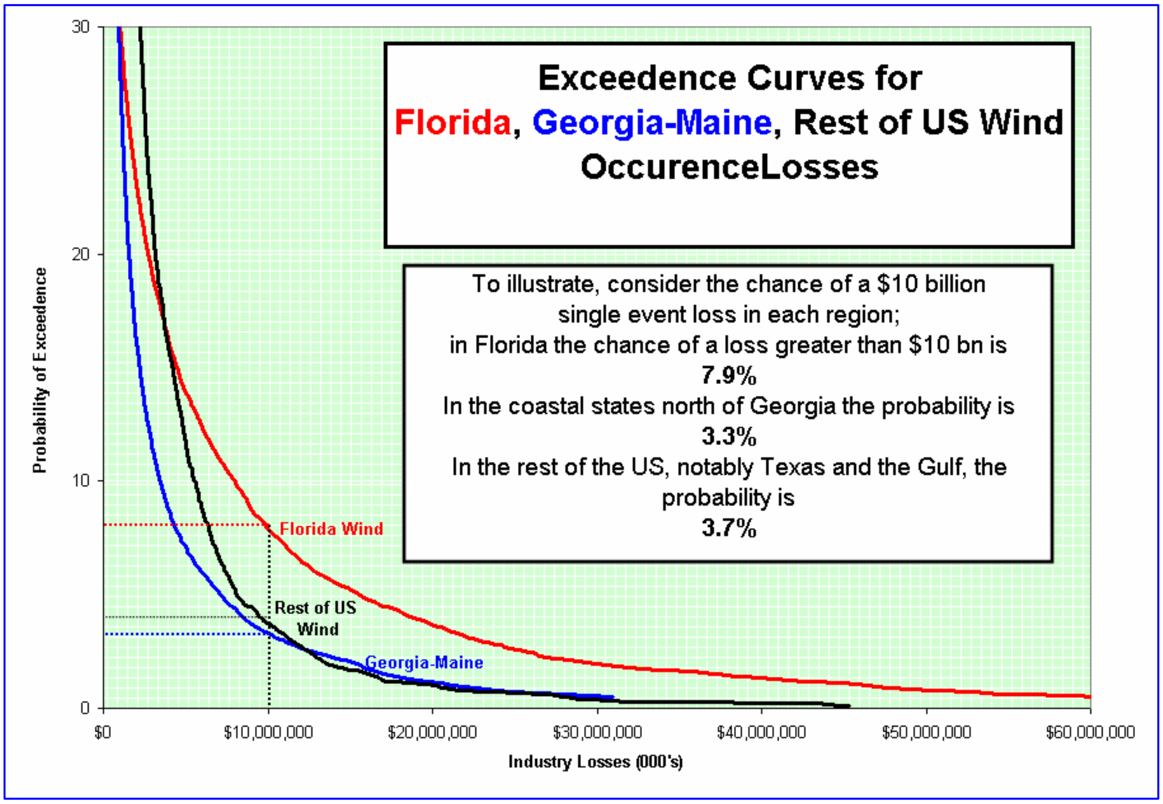
Name	Florida Wind	Georgia to Maine Wind	Rest US Wind	Nationwide Wind
Tag	FlaWnd	GtMWnd	OthWnd	NWnd
Linkages				NWnd
Type	ILW	ILW	ILW	ILW
Distribution	FlaWnd	GtMWnd	OthWnd	FlaWnd,GtMWnd,OthWnd
Region	US	US	US	US
Trigger Point				
Billions	Wind	Wind	Wind	Wind
$1.0				
$1.5				
$2.0				
$3.0				
$5.0	10	10		10
$7.5				
$10.0	10	10		10
$12.5	10	10		10
$15.0	10	10		10
$20.0	10	10	10	10
$25.0	10	10		10
$30.0	10	10		10
$40.0	10	10		10
$50.0	10	10		10

A $10 billion triggered ILW for Florida Wind could be written for a premium[8] of 17.5%. The example assumes that up to $25 million can be written, but if ceded, only $10 million is available. Similarly the 2% premium will be received if $25 million of $20 billion RUS is written, but only $10 million can be ceded. Note that there is a 10% transaction cost associated with either writing or ceding any deal selected. This can be viewed as combination of brokerage and/or bid-ask spread.

The Risk Environment

The risk environment in which the optimization decision must be taken is captured in the graphic below:

[8] To repeat footnote 6, the prices assumed herein are pre Katrina, as are the risk profiles.



The probability of events greater than $10 billion are respectively assumed to be 7.9%, 3.3% and 3.7% depending on whether the event is in FLA, GEO or RUS. Remember the corresponding premiums, at least for FLA and GEO, are 17.5% and 4.5%. Scenarios can be generated at any point on any of the curves. And the Nation Wide Curve can be generated from the component curves. The probability of exceeding a $10 billion event nationwide is 14.04%. We are assuming that there is no correlation between events hitting FLA, GEO or RUS. This is not necessary to the analysis; it is simplifying.

For example, one could have a $40 billion event hitting RUS and $10 billion event hitting FLA. (Sound familiar? How about Katrina and Wilma in 2005?) That scenario could be labeled j. Then all the deals i would have a $loss_{ij} = 0$ or 1 depending on whether the particular strike level was above or below the RUS $40 billion or the $10 billion FLA level. On the other hand the NWW covers would be triggered by the $10 billion[9] RUS level or by the $40 billion FLA loss, which ever came first, but it would be triggered in that scenario. In other words, while there is independence among our three principal zones, there is absolutely dependence in the NWW zones on the other three.

Risk Preferences

The risk preference constraints used are as follows:

1) Probability 30%, CVaR limit of 10% loss of risk capital

[9] We are assuming that all the ILW deals in this example are "one shot" without possibility of reinstatement.

2) Probability 20%, CVaR limit of 20% loss of risk capital
3) Probability 0.1%, CVaR limit of 100% loss of risk capital

The first of these might be articulated as "on the worst 30% of outcomes the expected net loss to capital should be no more than 10%". The second and third constraints are more stringent. The third says "even in those cases which will occur less than one in one thousand times (Probability <= 0.1%) the expected loss of capital is no more than 100%". This last put a constraint on losing all one's capital. It does not mean it cannot happen, it simply says that should not happen more frequently than once in one thousand years.

We talked extensively about CVaR in the previous paper so we will not repeat that, here. Suffice to say it is a tractable measure (i.e., it can be solved using linear programming) and it seems to capture the essence of most risk management's expressions of downside concern. It also has the virtue of being consistent with the way rating agencies view reinsurers.

The Solution

The model was run over 20,000 scenarios and the optimum (primal) solution is displayed in the table below and in the appended tables. The appended outputs correspond to many of the reports regularly generated from a ReAL solution. They were discussed in detail in the previous paper so will not be dwelt on here, but they can provide further explanation of the aspects of the solution to be discussed here.

Trigger (Billions)	Florida Wind	Georgia to Maine Wind	Rest US Wind	Nationwide Wind	Totals by Layer	Percentage
$1.0						
$1.5						
$2.0						
$3.0						
$5.0	$19.14			-$10.00	$9.14	5.52%
$7.5						
$10.0	$25.00				$25.00	15.08%
$12.5	$25.00				$25.00	15.08%
$15.0	$25.00				$25.00	15.08%
$20.0	$13.30	$10.00	$10.00		$33.30	20.09%
$25.0						
$30.0						
$40.0		$25.00			$25.00	15.08%
$50.0	$10.00	$13.30			$23.30	14.06%
Totals by Zone	$117.44	$48.30	$10.00	-$10.00	$165.74	
Percentage	70.86%	29.14%	6.03%	-6.03%		100.00%

The expected return on equity, before investment income, as shown in the appendix was 8.58%; initial capital being $100 million. The portfolio composition of the optimal solution is shown in the table above. Clearly, it is profitable to write a lot of FLA wind cover, in this case to a maximum exposure of $117.44 million. In particular the $10, $12.5, $15 and $50 billion trigger covers are written to their maximum. Fractions of what is possible are written in the $5 and $20 billion trigger covers, indicating that these make marginal contributions. About 30% of the exposure is written in GEO and a small amount is written in RUS. Noticeably a $10 NWW cover is ceded rather than assumed, which we return to examine momentarily.

For now, observe the following about the solution. First, it is diversified by zone. Second, it is diversified by layer. Third, it exposes capital to a maximum of $165.74 of aggregate limit. Finally, it cedes as well assumes risk. None of these features were required of the solution in the model specification. There was no forcing of diversification, as is typical of a "pillar" approach is some simulation solutions. There was no forcing of a layering, it chose to do that. There was no leverage limit imposed on the solution. In the solution, the leverage (maximum exposure divided by available capital) was an outcome of the interplay between the opportunity set and the specified risk preferences. All these features are testament to a robust model that produces realistic solutions, rather than forced answers.

This is particularly true for the ceding of risk. Prudent risk management in practice requires that underwriters be alive to the price of retrocessional opportunities as well as the price of risk to be assumed. Evidently, the internal logic of stochastic optimization has driven the model in this example to cede as well as assume risk. The total written premium for the optimum portfolio is $19 million. Of this the model has decided to spend $2.8 million on retro and buy protection for some aspects of its risk profile. The optimum solution purchases $10 million of the $5 billion NWW cover. Net written premium is therefore $16.2 million. After expensing transaction costs and expected losses, net of expected recoveries, the expected portfolio return is $8.58 million.

Before exploring the rationale of the retrocessional decision it is worth spending one more moment on understanding how the NWW will work. Given the other aspects of the solution it should be clear that a, say, $6 billion event in FLA will cause a loss due to the written FLA $5 billion cover, of $19.14 million, but that this will be offset by a recovery from the $5 billion NWW ILW of $10 million. If, however, the $6 billion event occurs in GEO or RUS then the written portfolio incurs no loss, but there is a recovery from the $5 NWW ILW. This is recovery without loss. It would not be allowed in traditional retrocession, but can occur here because of the industry aspects of ILWs. Given the potential of recovery from all possible scenarios, the expected recoveries is $2.95 million.

So, is the internal logic simply identifying an under-priced ILW where the premium is less than the expected recovery? It would seem so until transaction costs are taken into account. Remember these are set at 10% of premium, whether bought or sold. Therefore the cost of the cover is $2.8 million plus 10% for a total of $3.08 million. The model is actually entering into the purchase of a negative value transaction. It is spending $3.08 million to get an expected recovery of $2.95 million. In other words it expects to be out of pocket $0.13 million. And yet this is the optimal thing to do.

In observing the reinsurance industry, capital market participants are often curious about why reinsurers spend money of retrocession. Renaissance Re, ACE and XL, for example, average nearly 20% of Gross Written Premium spent on retrocession. Why does it make sense to purchase that amount at a negative spread? Why not just stick with the Gross Written Premium and save that money? The answer to that question lies in stated risk preferences. If the underwriter is indifferent to the risk profile he assumes then retrocession may not make sense. Once it is admitted that the portfolio profile matters, retrocession can make sense, and allocations of costs and capital can also be done on a risk adjusted basis.

Allocating the Retrocessional Cost on a conventional basis

The conventional method of allocating retrocession costs is to allocate by simple expected loss. We will refer to this as "a priori expected loss" since it uses the original scenario probabilities, unadjusted for risk preference. We assume that FLA GEO and

OPTIMAL PORTFOLIO COMPOSITION BY ZONE

(Expected returns calculated using original sample probabilities)

	FLA	GEO	RUS	NWW	TOTAL
Premium by zone	$18.02	$0.78	$0.20	-$2.80	$16.20
Transaction cost	$1.80	$0.08	$0.02	$0.28	$2.18
Net premium by zone	$16.21	$0.70	$0.18	-$3.08	$14.02
a priori Expected losses by zone	$8.08	$0.21	$0.10	-$2.95	$5.44
a priori Expected profit by zone	$8.14	$0.50	$0.08	-$0.13	$8.58
OR					
a priori Expected recoveries by zone	$1.31	$0.64	$1.00		
Allocating premium (by Exp. Recoveries)	$1.37	$0.66	$1.05		
Expected profit net of recoveries	$8.08	$0.47	$0.03		$8.58

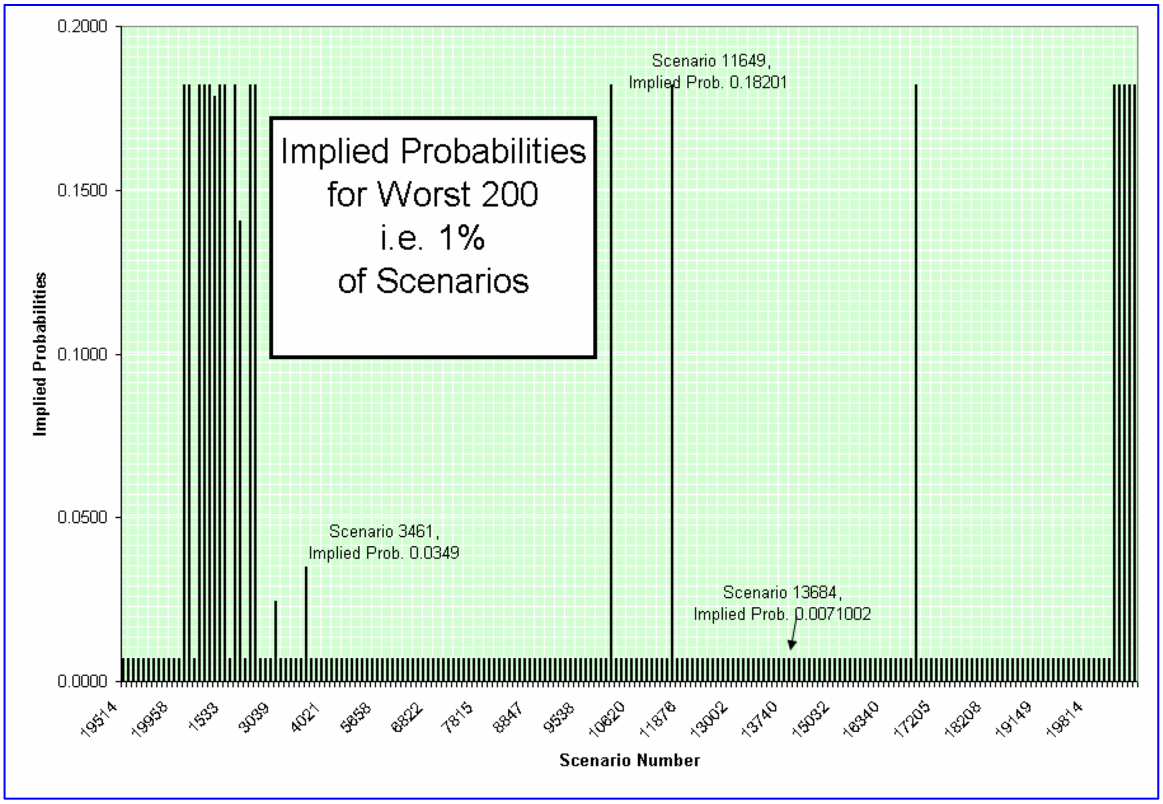
RUS are divisions of the company. The retro cover inures to all of them, so who should pay what? The conventional approach is laid out in the table above.

The premiums written in each zone are, respectively, $18.2 million, $0.78 million and $0.2 million in FLA GEO and RUS. After calculation of transaction costs and expected losses by zone the a priori profits by zone are, respectively, $8.14 million, $0.5 million and $0.08 million. And of course as we have already identified there is an expected net cost of $0.13 million that is spent on retrocession and had to be divvied up to the three zones.

One way to do that is by the zonal expected losses, in which case some 96% of costs would go to FLA, but that is far too crude. Instead, the expected recoveries can be calculated for each zone and they are listed in the allocation row as $1.31 million, $0.64 million and $1.0 million for FLA, GEO and RUS respectively. Premium and transaction costs can now be allocated by those proportions viz $1.37 million, $0.66 million and $1.05 million. Expected profits for each of the zones is now determined; 96.3% (=$8.08/$8.58) of the profits emanate from FLA even though it assumes only 70% of the limit.

Unfortunately, this allocation scheme takes no account of the risk preferences of the management. Instead it allocates as if management were indifferent to the various risky outcomes, even though the model has restricted the output to be consistent with management desires.

Implied Probabilities



As we have demonstrated in our previous paper the dual outputs from our optimizing model can be used to calculate implied probabilities and these carry with them the essence, the DNA so to speak, of risk preferences. The graphic above illustrates the nature of implied probabilities. It displays the worst 1% of the 20,000 scenarios used in our optimizing model, i.e. 200 scenarios.

In the a priori estimates of expected loss each scenario was given an equal probability of occurrence of 0.005% (=1/20,000). Implicit in the optimizing model is a vector of implied probabilities which accords different probabilities to each scenario, depending on its impact on risk preferences. In the graphic scenario 11,649 is given a probability of 0.18201, while scenario 19,684 is given a probability of 0.0071002. Evidently scenario 11,649 is one to avoid. The vector of implied probabilities is critical to explaining optimizing decisions within the model. It converts a priori analysis to risk adjusted analysis.

Decomposing the retrocessional cost on a risk adjusted basis

We can now repeat our conventional analysis on a risk adjusted basis. Essentially, the expected value calculations of the previous table are repeated using implied probabilities rather than a priori probabilities. The first feature to note in the table is that the “risk adjusted expected losses” for each zone are different from the a priori expected losses. This is true even before allocation of any retrocessional cost. The risk adjusted expected losses are $15.43 million, $0.60 million and $0.18 million, respectively. This

OPTIMAL PORTFOLIO COMPOSITION BY ZONE (Risk Adjusted Basis)

(Risk adjusted expected amounts use the implied probability vector)

	FLA	GEO	RUS	NWW	TOTAL
Premium by zone	$18.02	$0.78	$0.20	-$2.80	$16.20
Transaction cost	$1.80	$0.08	$0.02	$0.28	$2.18
Net premium by zone	$16.21	$0.70	$0.18	-$3.08	$14.02
Risk Adjusted Expected losses by zone	$15.43	$0.60	$0.18	-$3.72	$12.49
Risk Adjusted Expected profit by zone	$0.78	$0.11	$0.00	$0.64	$1.53
OR					
Risk Adj. Expected recoveries by zone	$2.14	$0.64	$0.95		
Allocating premium (by RAERecoveries)	$1.77	$0.53	$0.78		
Expected profit net of recoveries	$1.15	$0.22	$0.16		$1.53

leaves risk adjusted expected profits of $0.78 million, $0.11 million and $0.00 million. This is a much less skewed sourcing of expected profits than in the a priori case where some 96.3% of profits came from FLA deals. Indeed the real surprise is perhaps that on a risk adjusted basis, the $5 billion NWW retrocessional cover is almost as valuable as the FLA covers. As before the cost of obtaining the retro is $3.08 million. The risk adjusted value of the recoveries gained from this cover is $3.74 million for expected profit of $0.64 million. In other words, on a risk adjusted basis, the retro is almost as valuable as the FLA writes.

We can now allocate the premium to the zones. Risk adjusted expected recoveries are, respectively, $2.14% million, $0.64 million and $0.95 million. The proportions are 57% (=$2.14/$3.72), 17% and 25%. These fractions are used to allocate the premium plus the transaction costs. The result is zonal costs of $1.77 million, $0.53 million and $0.78 million. Profits by zone are $1.15 million, $0.22 million and $0.16 million. These amounts are proportionately very close to the proportion of the limits by zone.

Allocating Retrocessional Premium to Zones

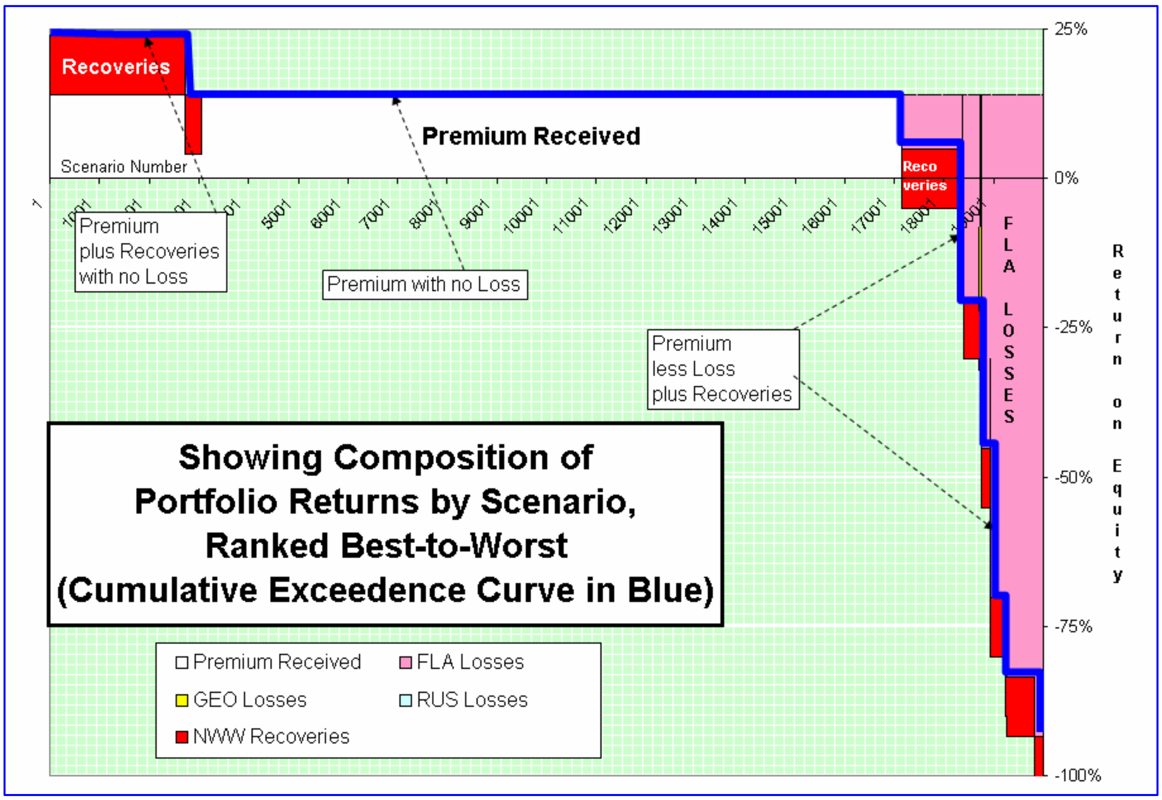
	FLA	GEO	RUS	Retro Cost
Proportionate to Exp. Loss	44%	22%	34%	$3.08
Proportionate to Risk Adj. Exp. Loss	57%	17%	25%	$3.08

How different is the allocation of the retro premium? That is captured in the preceding comparison table. When allocating on the basis of a priori expected recoveries, some 44% of the premium cost is allocated to the FLA division. However, on a risk adjusted basis that allocation should be more like 57% of the costs, with correspondingly less charged to the GEO and RUS divisions.

In dollar terms this may not seem like a big deal. The allocation of the retro should be $1.77 million vs. the $1.37 million conventionally allocated. It will still leave the expected profits from FLA dominating the profits from the portfolio as a whole. However, what is suggested here is that that may not be the best way to view the results. Instead it should be done on a risk adjusted basis.

Viewing the Contributions from each Zone

Many sports teams are built around star players. They score the most points and consistently are rated as MVP and match winners. But for every star like Michael Jordan there are others who come with their own set of problems, both on and off the field. As team owners and management try to project wholesome images and pure competitive superiority, stars can undermine the image with their own unsavory antics, while still being their team's highest point scorer. Team owners and managements have to decide



how much bad star they are prepared to take, in order to retain the good star. That depends on their (or the league's) preferences.

So it is with reinsurance and retrocession.

FLA clearly puts the premium dollars on the bottom line. However, it comes with risks of huge losses. It might appear that the FLA contributes 96% of the profits of the portfolio, but, given other preferences and objectives the risk adjusted contribution is closer to 75%. This is still huge, but it is a more nuanced view of the contribution. By the same token the contributions of GEO and RUS are elevated. A priori they look anemic. Risk adjusted, they clearly fulfill an important role.

We can also try to view the contributions graphically. The diagram above illustrates both the cumulative exceedence curve (blue line) and attempts to show the component losses by zone. Remember, the blue line is the net curve of the optimal portfolio. The contributions to loss are dominated by the shaded red areas where FLA contributes to loss. GEO and RUS also cause losses but the shadings for them are hardly visible.

The other feature of the graph that is instructive is the area of heavy red. This represents recoveries from NWW retro purchase. Notice that the retro recoveries make a contribution at nearly all loss level scenarios (and some profit scenarios). Red moves the blue line upwards and to the right. Retrocessional capital is valuable. As we will see it may be more valuable than real capital, simply because real capital may only relieve a single or a few binding risk capital constraints. To see this it is worth asking how the existing capital should be allocated to the zones. Like the retro allocation it will involve risk adjusted returns, but it will also require a more explicit calculation of the binding risk preference constraints.

On Capital Allocation

We proceed with a series of observations.

Our first observation is, if we had to allocate capital based upon the worst possible scenario, we would do it on the basis of the total limits, i.e., $117.44 for FLA, $48.30 for GEO and $10 for RUS for a total of $165.74. (In the tables below I have allocated the purchase of NWW on a risk adjusted basis as described earlier. We get the same allocation either before or after its allocation.) We don't have $165.74 of capital, only $100. That is what we are allocating. And so we could just scale the proportions, but that is not correct. Our risk preference constraints speak to probable events, and we know from the solution that two are binding – 20% and 0.1%. (See the risk report in the numerical appendix.) We also know the relative dual values of each of them 0.67 and 41.09.

Second observation, we can calculate the risk adjusted expected loss (i.e. using implied probabilities) at each of the binding CVaR constraints – see the table below.

		FLA	GEO	RUS	NWW	Premium		DUAL
Worst 20% of Scenarios								
Implied on worst 20%	28.6%							
Risk Adj. Expected Loss		$53.87	$1.79	$0.31	$8.32	$14.02	-34%	**0.67**
RAEL Allocation of Retro		$0.96	$0.03	$0.01				
		$52.91	$1.75	$0.30				
		96.3%	3.2%	0.6%				
Worst 0.1% of Scenarios								
Implied on worst 0.1%	1.0%							
Risk Adj. Expected Loss		$109.74	$29.32	$1.78	$10.00	$14.02	-117%	**41.09**
RAEL Allocation of Retro		$0.78	$0.21	$0.01				
		$108.96	$29.11	$1.77				
		77.9%	20.8%	1.3%				
Absolute Worst Scenario								
Maximum		$117.44	$48.30	$10.00	$10.00	$14.02	-141.72%	-
		$0.67	$0.27	$0.06				
		$116.77	$48.03	$9.94				
		66.8%	27.5%	5.7%				

In the table, for example, at the 20% constraint, 96.3% of the total risk adjusted expected losses emanate from FLA. At the 0.1% scenario, of the risk adjusted expected losses, 77.9% of total risk adjusted expected losses emanate from FLA. And of course, in the worst case 66.8% come from FLA. Since we are not concerned with the absolute worst case, the actual capital allocation should be a linear combination of the two binding constraints. (I have ignored the 30% constraint because it is not binding.)

Third observation, the linear combination should be relative to the duals. They express the "bindingness" of the constraints (if that's a word). So the next two tables show the affect of that.

		Realtive Dual Value
20% Dual	0.67	1.6%
0.1% Dual	41.09	98.4%
	41.76	100.0%

The correct way to weight the relative allocations is now clear. In percentage terms the correct FLA allocation is (1.6%) of 96.3% plus (98.4%) of 77.9% = 79.5%. The

	FLA	GEO	RUS	NWW	Premium	DUAL
Risk Adjusted Capital Allocation	79.5%	20.9%	1.3%			
RACA Dollar Basis	$79.5	$20.9	$1.3			

same reasoning gives 20.9% to GEO and 1.3% to RUS. In many ways this is a satisfactory solution because it acknowledges that the least profitable zone (RUS) still requires capital. Many capital allocation schemes often attribute zero capital to a unit that produces zero risk adjusted marginal return while still being part of the solution.

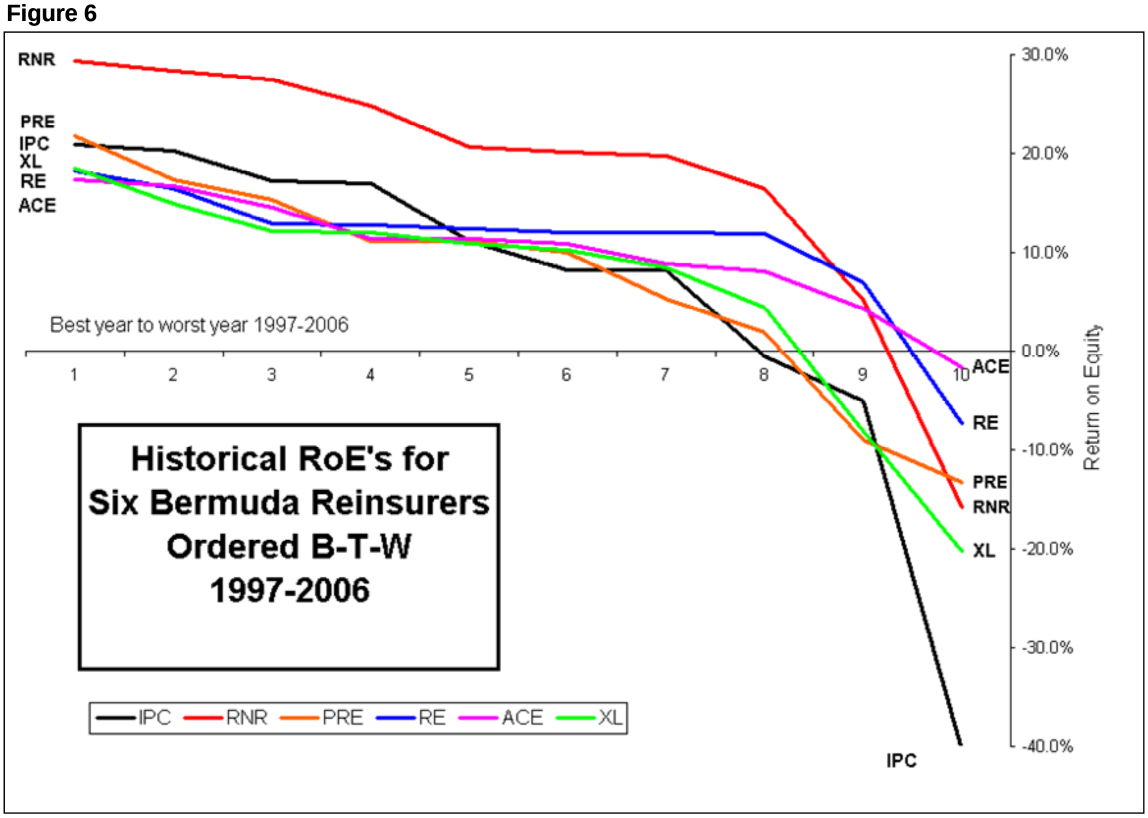
Conclusion and empirics

This paper has tried to show how the internal workings of an optimization approach to underwriting portfolio selection can be used to gain insight into the workings of allocation processes. Optimal allocations, whether of retro cost or existing capital, improperly viewed can lead to suboptimal decision making. The proper allocations rest heavily on implied probabilities and dual values. Together these convey the juxtaposition of risk preference and opportunity. We have focused on one way of expressing preference (CVaR) but we believe the results carry through equally well for VaR or any other expressions of preference, such as quadratic, etc. (although the algebra might quite problematic).

Simply put, it seems safe to say that it is impossible to answer questions about allocation without knowing what risk preferences are.

This raises the question of why such little emphasis is put on gleaning the risk preferences of existing reinsurance companies. It is, to be sure, a difficult exercise but worthwhile it would seem.

Figure 6



In an attempt to glean insight into reinsurer's preferences consider the graphic above. It displays the performance of six public reinsurers over the last ten years. The results are ordered in such a way (from best to worst) to emulate the exceedence curves we displayed for the three-zone example. To us it suggests that these reinsurers have both different performance and different appetites for risk. In future papers we will explore these empirical differences as well as lay out more detail on capital allocation formulas.

Demonstrating the virtues of marginal analysis from optimization techniques, we believe can reveal other properties of optimal decision making beyond those discussed in our first two papers. It can lead to powerful insights and to the furtherance of good practice. Also it can lead to powerful avenues for further inquiry into actual practice. That however is for a third paper and is not discussed here.

REFERENCES

Avellaneda, Marco and Peter Laurence. 2000. *Quantitative Modeling of Derivative Securities, From Theory to Practice*, Chapman and Hall, Boca Raton.

Claessens, Stijn and Jerome Kreuser. 2004. “A Framework for Strategic Foreign Reserves Risk Management,” in *Risk Management for Central Bank Foreign Reserves*, European Central Bank Publication, Frankfurt am Main, May.

Christensen, Claus Vorm. 2000 Implied Loss Distributions for Catastrophe Insurance Derivatives. working paper, Laboratory of Acturial Mathematics, University of Copenhagen (Available at: http//www.imf.au.dk/~vorm)

Jarrow, Robert A. 2005. “Valutaion of Insurance Contracts in Incomplete Markets”, Prelimianry Paper, Cornell University, April 25.

Jorion, Philippe. 2000. *Value at Risk*, Second Edition, McGraw-Hill.

Kreuser, Jerome and Morton Lane. An Introduction to the Benefits of Optimization Models for Underwriting Portfolio Selection, *Proceedings of the 28th International Congress of Actuaries*, Paris, June 2006.

Lane Morton N. 1995-2007 Various papers on developments in the Insurance Linked Securitization (Cat Bond) market, including annual review papers are available on, http//www.lanefinancialllc.com

Lane Morton N. (Ed) 2002 *Alternative Risk Strategies,* Risk Books

Lane, Morton and Jerome Kreuser. 1980. "Designing Investment Strategies for Fixed-Income Portfolios," in *Extremal Methods and Systems Analysis*, Ed. A.V. Fiacco and K. O. Kortanek, Springer-Verlag, New York.

Lane Morton and O. Y. Movchan. 1998 The Perfume of the Premium II *Sedgwick Lane Financial Trade Notes* (Available at: http//www. lanefiancialllc.com)

McNeil, Alexander J., Rüdiger Frey and Paul Embrechts, *Quantitative Risk Management*, Princeton University Press, 2005.

Rockafellar, R. Tyrrell and Stanislav Uryasev. 2002. "Conditional Value-at-Risk for General Loss Distributions," in the *Journal of Banking and Finance*, Vol. 26(7), July.

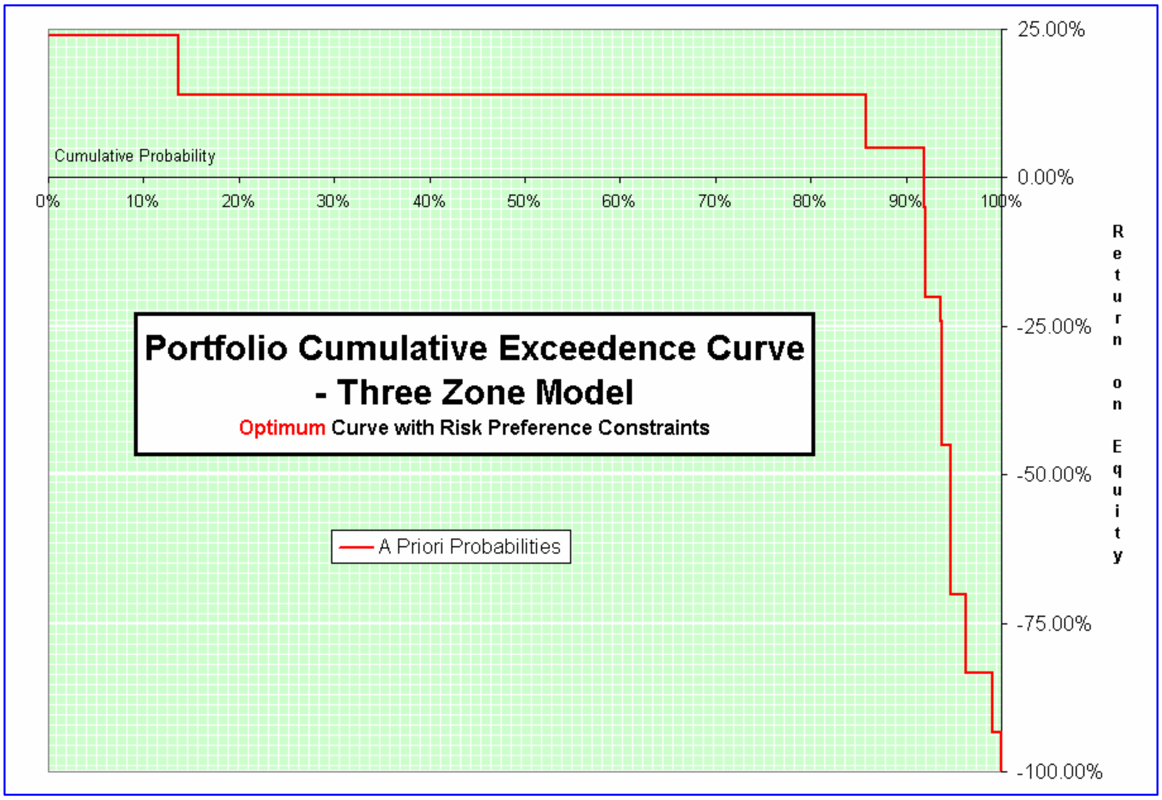
Theiler, Ursula and Vladimir Bugera, Alla Revenko, Stanislav Uryasev. 2002. Regulatory Impacts on Credit Portfolio Management.

Uryasev, Stanislav. 2000. "Conditional Value-at-Risk: Optimization Algorithms and Applications," *Financial Engineering News*, Issue, February

Ziemba, William T. and John M. Mulvey. 1998, *Worldwide Asset and Liability Modeling*, Cambridge University Press, UK

NUMERICAL APPENDIX

The following Pages contain full details of the numerical example Optimum Portfolio Return Profile and Portfolio Detail.



Trigger (Billions)	Florida Wind	Georgia to Maine Wind	Rest US Wind	Nationwide Wind	Totals by Layer	Percentage
$1.0						
$1.5						
$2.0						
$3.0						
$5.0	$19.14			-$10.00	$9.14	5.52%
$7.5						
$10.0	$25.00				$25.00	15.08%
$12.5	$25.00				$25.00	15.08%
$15.0	$25.00				$25.00	15.08%
$20.0	$13.30	$10.00	$10.00		$33.30	20.09%
$25.0						
$30.0						
$40.0		$25.00			$25.00	15.08%
$50.0	$10.00	$13.30			$23.30	14.06%
Totals by Zone	$117.44	$48.30	$10.00	-$10.00	$165.74	
Percentage	70.86%	29.14%	6.03%	-6.03%		100.00%

Expected Ending Balance Sheet (Millions $US)		
Assets		
		$108.58
Liabilities		
Initial Capital	$100.00	
Expected Retained Earnings	$8.58	
Capital and Surplus		$108.58
Expected Income statement		
Income		
Written Premium	$19.00	
Ceded Premium	-$2.80	
Net written Premium		$16.20
Investment Income		$0.00
		$16.20
Expenses		
Expected Losses	-$8.39	
Expected Recoveries	$2.95	
Expected Net Losses		-$5.44
Brokerage & Acquisition expense		-$2.18
General & Administrative		$0.00
Expected Profit		$8.58
Expected Rate on Equity		8.58%
Exposure Report		
Total Net Exposure		$155.74
Net Premium to Net Cover		0.10
Leverage: Exposure/Capital		1.56
Underwriting Report		
Premiums Written		$19.00
Premiums Ceded		-$2.80
Expected Losses		-$8.39
Expected Recoveries		$2.95
Expected Underwriting Profit		$10.76
Portfolio Loss Ratios		
Net Written Premiums		$16.20
Expected Net Losses		$5.44
Expected Net Loss Ratio		33.57%

ILW — Florida Wind

Trigger (Billiions)	Available Price (Pre-bid/ask)	Threshold Price	Available Cover Limit	Optimal Solution	Cover Marginal	Retro Threshold Price	Available Retro Limit	Optimal Retro	Retro Marginal
$1.0									
$1.5									
$2.0									
$3.0									
$5.0	26.00%	26.00%	25	**$19.14**		20.45%	10		-6.11%
$7.5									
$10.0	17.50%	15.99%	25	**$25.00**	1.36%	12.53%	10		-5.47%
$12.5	15.00%	13.41%	25	**$25.00**	1.43%	10.49%	10		-4.96%
$15.0	13.00%	12.03%	25	**$25.00**	0.88%	9.43%	10		-3.93%
$20.0	9.50%	9.50%	25	**$13.30**		7.47%	10		-2.23%
$25.0	7.20%	7.47%	25		-0.24%	5.88%	10		-1.45%
$30.0	5.80%	6.21%	25		-0.37%	4.90%	10		-0.99%
$40.0	4.50%	5.20%	25		-0.63%	4.11%	10		-0.43%
$50.0	4.00%	4.00%	25	**$10.00**		3.15%	10		-0.94%

ILW — Georgia to Maine Wind

Trigger (Billiions)	Available Price (Pre-bid/ask)	Threshold Price	Available Cover Limit	Optimal Solution	Cover Marginal	Retro Threshold Price	Available Retro Limit	Optimal Retro	Retro Marginal
$1.0									
$1.5									
$2.0									
$3.0									
$5.0	8.00%	10.09%	25		-1.88%	8.00%	10		
$7.5									
$10.0	4.50%	5.52%	25		-0.91%	4.37%	10		-0.14%
$12.5	4.30%	4.70%	25		-0.36%	3.71%	10		-0.65%
$15.0	3.70%	3.92%	25		-0.19%	3.09%	10		-0.68%
$20.0	2.80%	2.80%	25	**$10.00**		2.20%	10		-0.66%
$25.0	2.00%	2.00%	25		0.00%	1.57%	10		-0.47%
$30.0	1.80%	1.82%	25		-0.02%	1.43%	10		-0.41%
$40.0	1.50%	0.91%	25	**$25.00**	0.53%	0.70%	10		-0.88%
$50.0	1.00%	1.00%	25	**$13.30**		0.79%	10		-0.23%

ILW — Rest US Wind

Trigger (Billiions)	Available Price (Pre-bid/ask)	Threshold Price	Available Cover Limit	Optimal Solution	Cover Marginal	Retro Threshold Price	Available Retro Limit	Optimal Retro	Retro Marginal
$1.0									
$1.5									
$2.0									
$3.0									
$5.0									
$7.5									
$10.0									
$12.5									
$15.0									
$20.0	2.00%	2.00%	25	**$10.00**		1.57%	10		-0.47%
$25.0									
$30.0									
$40.0									
$50.0									

ILW — Nationwide Wind

Trigger (Billiions)	Available Price (Pre-bid/ask)	Threshold Price	Available Cover Limit	Optimal Solution	Cover Marginal	Retro Threshold Price	Available Retro Limit	Optimal Retro	Retro Marginal
$1.0									
$1.5									
$2.0									
$3.0									
$5.0	28.00%	43.69%	25		-14.12%	34.85%	10	**$10.00**	7.54%
$7.5									
$10.0	20.00%	23.64%	25		-3.27%	18.71%	10		-1.42%
$12.5	17.00%	19.28%	25		-2.05%	15.23%	10		-1.95%
$15.0	14.50%	16.58%	25		-1.87%	13.10%	10		-1.54%
$20.0	11.00%	12.81%	25		-1.63%	10.13%	10		-0.96%
$25.0	8.70%	9.92%	25		-1.10%	7.84%	10		-0.95%
$30.0	7.70%	8.16%	25		-0.41%	6.43%	10		-1.40%
$40.0	5.50%	6.50%	25		-0.90%	5.14%	10		-0.39%
$50.0	4.50%	4.85%	25		-0.31%	3.83%	10		-0.74%

RANKING THE WRITES						
Zone	Trigger (Billiions)	Available Price (Pre-bid/ask)	Threshold Price	Available Cover Limit	Optimal Solution	Cover Marginal
FLA	$12.5	15.00%	13.41%	25	$25.00	1.43%
FLA	$10.0	17.50%	15.99%	25	$25.00	1.36%
FLA	$15.0	13.00%	12.03%	25	$25.00	0.88%
GEO	$40.0	1.50%	0.91%	25	$25.00	0.53%
GEO	$50.0	1.00%	1.00%	25	$13.30	0.00%
RUS	$20.0	2.00%	2.00%	25	$10.00	0.00%
GEO	$20.0	2.80%	2.80%	25	$10.00	0.00%
FLA	$50.0	4.00%	4.00%	25	$10.00	0.00%
FLA	$20.0	9.50%	9.50%	25	$13.30	0.00%
FLA	$5.0	26.00%	26.00%	25	$19.14	0.00%
GEO	$25.0	2.00%	2.00%	25		0.00%
GEO	$30.0	1.80%	1.82%	25		-0.02%
GEO	$15.0	3.70%	3.92%	25		-0.19%
FLA	$25.0	7.20%	7.47%	25		-0.24%
NWW	$50.0	4.50%	4.85%	25		-0.31%
GEO	$12.5	4.30%	4.70%	25		-0.36%
FLA	$30.0	5.80%	6.21%	25		-0.37%
NWW	$30.0	7.70%	8.16%	25		-0.41%
FLA	$40.0	4.50%	5.20%	25		-0.63%
NWW	$40.0	5.50%	6.50%	25		-0.90%
GEO	$10.0	4.50%	5.52%	25		-0.91%
NWW	$25.0	8.70%	9.92%	25		-1.10%
NWW	$20.0	11.00%	12.81%	25		-1.63%
NWW	$15.0	14.50%	16.58%	25		-1.87%
GEO	$5.0	8.00%	10.09%	25		-1.88%
NWW	$12.5	17.00%	19.28%	25		-2.05%
NWW	$10.0	20.00%	23.64%	25		-3.27%
NWW	$5.0	28.00%	43.69%	25		-14.12%

RANKING THE RETROS

Zone	Trigger (Billiions)	Retro Threshold Price	Available Retro Limit	Optimal Retro	Retro Marginal
NWW	$5.0	34.85%	10	$10.00	7.54%
GEO	$5.0	8.00%	10		0.00%
GEO	$10.0	4.37%	10		-0.14%
GEO	$50.0	0.79%	10		-0.23%
NWW	$40.0	5.14%	10		-0.39%
GEO	$30.0	1.43%	10		-0.41%
FLA	$40.0	4.11%	10		-0.43%
GEO	$25.0	1.57%	10		-0.47%
RUS	$20.0	1.57%	10		-0.47%
GEO	$12.5	3.71%	10		-0.65%
GEO	$20.0	2.20%	10		-0.66%
GEO	$15.0	3.09%	10		-0.68%
NWW	$50.0	3.83%	10		-0.74%
GEO	$40.0	0.70%	10		-0.88%
FLA	$50.0	3.15%	10		-0.94%
NWW	$25.0	7.84%	10		-0.95%
NWW	$20.0	10.13%	10		-0.96%
FLA	$30.0	4.90%	10		-0.99%
NWW	$30.0	6.43%	10		-1.40%
NWW	$10.0	18.71%	10		-1.42%
FLA	$25.0	5.88%	10		-1.45%
NWW	$15.0	13.10%	10		-1.54%
NWW	$12.5	15.23%	10		-1.95%
FLA	$20.0	7.47%	10		-2.23%
FLA	$15.0	9.43%	10		-3.93%
FLA	$12.5	10.49%	10		-4.96%
FLA	$10.0	12.53%	10		-5.47%
FLA	$5.0	20.45%	10		-6.11%

Optimal Risk Report for problem: Multi zone wind. Generated on 08/13/05 and time 17:10:11

Capital (Million)	100
Transaction Cost%	10.00%
Bid/Ask Spread%	0.00%
Investment Return%	0.00%
Retro Limit as %Capital	50.00%

Problem Class	Distribution Based
Model Name	basicmod.gms

Risk Actual, Limits, and Technical Data

Prob%	VaR %Loss	CVaR %Loss	Limit CVaR	Alpha	Dual	
30.00%	14.02%	-8.66%	-10.00%	-11.47	0.00	
20.00%	14.02%	-20.00%	-20.00%	-14.02	0.67	Binding
0.10%	-93.42%	-100.00%	-100.00%	93.42	41.09	Binding

Other Solution Information and VaR and CVaR Values

Written Premium	19.00
Ceded Premium	2.80
Expected Retained Earnings	8.58
Marginals on Retrocession Maximum	0.00
Risk adjusted marginal return on capital	6.78%
Marginal return per dollar of non-risk capital	17.47%

VaR and CVaR Values along the Distribution

Prob%	VaR %Loss(-)	CVaR %Loss(-)
100.00%	24.02%	8.58%
90.00%	24.02%	6.87%
80.00%	14.02%	5.52%
70.00%	14.02%	4.30%
60.00%	14.02%	2.68%
50.00%	14.02%	0.41%
40.00%	14.02%	-2.99%
30.00%	14.02%	-8.66%
20.00%	14.02%	-20.00%
10.00%	4.88%	-50.16%
9.00%	4.88%	-56.27%
8.00%	-20.12%	-63.71%
7.00%	-20.12%	-69.94%
6.00%	-45.12%	-77.20%
5.00%	-70.12%	-82.12%
4.00%	-70.12%	-85.12%
3.00%	-83.42%	-86.77%
2.00%	-83.42%	-88.45%
1.00%	-83.42%	-93.48%
0.90%	-93.42%	-94.15%
0.80%	-93.42%	-94.24%
0.70%	-93.42%	-94.36%
0.60%	-93.42%	-94.52%
0.50%	-93.42%	-94.74%
0.40%	-93.42%	-95.07%
0.30%	-93.42%	-95.61%
0.20%	-93.42%	-96.71%
0.10%	-93.42%	-100.00%
0.09%	-93.42%	-100.73%
0.08%	-93.42%	-101.65%
0.07%	-93.42%	-102.82%
0.06%	-93.42%	-104.39%
0.05%	-93.42%	-106.58%
0.04%	-93.42%	-109.87%
0.03%	-93.42%	-115.35%
0.02%	-103.42%	-123.82%
0.01%	-131.72%	-136.72%
5.0E-05	-141.72%	-141.72%
0.00%	-141.72%	-141.72%

MATHEMATICAL APPENDIX: Algebra of Allocations

In this appendix we develop the algebra for allocations for recoveries and for capital allocation under some simplifying assumptions. We do this using the dual equations of our model.

1. **The Dual Problem**

We write the dual relations based on the model. The model equations are replicated in the following and each dual variable is identified with its associated constraint.

$$\max_{\substack{funds0,\ funds1_i, z_i, \alpha_k, \\ deal_j, retro_j, losses_i, gains_i}} \sum_{i \in I} \overline{\rho}_i\, funds1_i$$

$$funds0 - \sum_{j \in J} \left(p_j (1 - \overline{trs})(1 - \overline{bda}) deal_j - p_j (1 + \overline{trs})(1 + \overline{bda}) retro_j \right) = 0 \qquad [\text{Dual: } u_0]$$

$$funds1_i - \left(1 + \overline{rate}\right) funds0 + losses_i - gains_i = \overline{rate} \times \overline{capital} \qquad [\text{Dual: } u_i]$$

$$losses_i - \sum_{j \in J} deal_j \overline{loss_{i,j}} = 0 \qquad [\text{Dual: } ul_i]$$

$$gains_i - \sum_{j \in J} retro_j \overline{loss_{i,j}} = 0 \qquad [\text{Dual: } ug_i]$$

$$\sum_{j \in J} retro_j \le \overline{tm} \times \overline{capital} \qquad [\text{Dual: } ucap]$$

$$-funds1_i - \alpha_k - z_{k,i} \le 0 \qquad [\text{Dual: } uz_{k,i}]$$

$$\sum_{i \in I} \overline{\rho}_i z_{k,i} + \left(1 - \overline{c}_k\right) \alpha_k \le \left(1 - \overline{c}_k\right) \overline{cvar_k} \times \overline{capital} \qquad [\text{Dual: } ucvar_k]$$

Bounds on deals and non-negativity constraints

$$0 \le deal_j \le \overline{\text{deal limit}_{\text{j}}} \qquad [\text{Dual: } ubd_j]$$

$$0 \le retro_j \le \overline{\text{retro limit}_{\text{j}}} \qquad [\text{Dual: } ubr_j]$$

$$-z_{k,i} \le 0 \qquad [\text{Dual: } uzl_{k,i}]$$

$funds0,\ funds1_i, \alpha_k$ can otherwise take any value.

The dual objective is:

(1) $$\min_{\substack{u_0, u_i, ul_i, ug_i, ucap \\ ucvar_k, uz_{k,i}, ubd_j, ubr_j uzl_{k,i}}} \sum_{k \in K} (1-\bar{c}_k)\overline{cvar_k} \times \overline{capital} \times ucvar_k + \overline{rate} \times \overline{capital} \sum_{i \in I} u_i + \overline{tm} \times \overline{capital} \times ucap + \sum_{j \in J} ubd_j + \sum_{j \in J} ubr_j$$

Corresponding to the $funds0$ variable, we have:

(2) $$0 = u_0 - (1+\overline{rate})\sum_{i \in I} u_i$$

Corresponding to the $funds1_i$ variables, we have:

(3) $$\bar{\rho}_i = u_i - \sum_{k \in K} uz_{k,i}$$

Corresponding to the $deal_j$ variables, we have:

(4) $$0 = -p_j(1-\overline{trs})(1-\overline{bda})u_0 - \sum_i \overline{loss}_{i,j}\ ul_i + ubd_j$$

Corresponding to the $retro_j$ variables, we have:

(5) $$0 = p_j(1+\overline{trs})(1+\overline{bda})u_0 - \sum_i \overline{loss}_{i,j}\ ug_i + ucap + ubr_j$$

Corresponding to the $losses_i$ and $gains_i$ variables, we have:

(6) $$0 = u_i + ul_i$$
$$0 = -u_i + ug_i$$

Corresponding to the α_k variables, we have:

(7) $$0 = -\sum_{i \in I} uz_{k,i} + (1-\bar{c})ucvar_k$$

Corresponding to the $z_{k,i}$ variables, we have:

(8) $$0 \le -uz_{k,i} + \bar{\rho}_i ucvar_k$$

We have the following constraints on the dual variables:

(9) $$0 \le \begin{matrix} uone_j \\ ucap \\ uz_{k,i} \\ ucvar_k \end{matrix}$$

with all the others being unbounded.

Lastly, we have the primal/dual equality at the optimum:

(10) $$\sum_{i \in I} \bar{\rho}_i \, funds1_i = \sum_{k \in K} (1 - \bar{c}_k) \overline{cvar_k} \times \overline{capital} \times ucvar_k + \overline{rate} \times \overline{capital} \sum_{i \in I} u_i + \overline{tm} \times \overline{capital} \times ucap + \sum_{j \in J} ubd_j + \sum_{j \in J} ubr_j$$

We omit writing down the complementarity conditions.

2. <u>Implied Probabilities and Pricing</u>

If we rewrite equation (2), we get:

(11) $$\frac{u_0}{(1 + \overline{rate})} = \sum_{i \in I} u_i$$

Now, let:

(12) $$\tilde{u}_i = \frac{u_i}{\sum_{i \in I} u_i} = \frac{(1 + \overline{rate}) u_i}{u_0}$$

As these sum to one, we will call them "implied probabilities" for the scenarios.

As we will see, these implied probabilities have a similar interpretation and use as in the Arrow-Debreu "risk-neutral probabilities" or "risk-adjusted probabilities" and are used to compute "state-price deflators" using $\overline{rate}$ as the riskless rate. The $\tilde{u}_i$ define the relative price of a “state” and identify those states that are most important in terms of impacting risk capital.

The losses, $\overline{loss}_{i,j}$, for scenario *i* and deal *j* are the unit losses. The net return per unit of written premium of the deal then is $p(1-\overline{trs})-\overline{loss}_i$ for scenario *i* where we drop the index *j*.
For written premium, the expected return per unit of premium is then $\sum_{i\in I}\overline{\rho}_i\left(\overline{price}(1-\overline{trs})-\overline{loss}_i\right)=\overline{price}(1-\overline{trs})-\sum_{i\in I}\overline{\rho}_i\overline{loss}_i$.

Now, let us compute the expected return on writing the deal in terms of the risk neutral probabilities. In the remainder, let us assume for simplicity that $\overline{rate}=0$. Therefore, the risk adjusted expected return on the deal with our implied probabilities is:

(13)
$$
\begin{aligned}
\sum_{i\in I}\left(\overline{price}(1-\overline{trs})-\overline{loss}_i\right)\tilde{u}_i &= \frac{1}{u_0}\sum_{i\in I}\left(\overline{price}(1-\overline{trs})-\overline{loss}_i\right)u_i \\
&\text{from (12)} \\
&= \frac{1}{u_0}\left(\overline{price}(1-\overline{trs})u_0-\sum_{i\in I}\overline{loss}_i u_i\right) \\
&\text{from (11)} \\
&= \frac{1}{u_0}\left(\overline{price}(1-\overline{trs})u_0+ubd-\overline{price}(1-\overline{trs})u_0\right) \\
&\text{from (4) and (6)} \\
&= \frac{ubd}{u_0}
\end{aligned}
$$

This then gives us the following fundamental equation for all pricing issues.

(14)
$$
\overline{price}(1-\overline{trs})-\sum_{i\in I}\overline{loss}_i\tilde{u}_i=\frac{ubd}{u_0}
$$

The equations for computing threshold prices derived from these are given in our previous paper and are not repeated here.

3. **<u>The marginal return to risk capital</u>**

Now we will review the main pricing equation in some more detail.
We have for any price *j*:

(15) $$\sum_{i\in I}\left(\overline{price}_j(1-\overline{trs})-\overline{loss}_{i,j}\right)\tilde{u}_i=\frac{ubd_j}{u_0}$$

where u_0 is the "marginal return per dollar of non-risk capital". In our example $u_0 = 1.1747$.

Now let:

(16) $$pl_{i,j}=\overline{price}_j(1-\overline{trs})-\overline{loss}_{i,j}$$

and define R_j as:

(17) $$R_j=\sum_{i\in I} pl_{i,j}\tilde{u}_i$$

and we have that R_j is called the risk-adjusted expected profit for deal *j*.

And so:

(18) $$R_j=\frac{ubd_j}{u_0}$$

Let us discuss what equation (18) means. First, u_0 is the marginal return on a unit of non-risk capital. This is capital "inserted" into the equation (9) (primal model) on the right-hand side; or equivalently at the beginning of the year as it changes the value of the variable $funds0$. It is more valuable than risk capital because it can be used in any state constraint and the constraint on risk capital may call for a percentage of risk capital (for example for a 50% loss of capital) so, in essence, risk capital may be diluted. If the only risk constraint we had in the model was one at 100% of capital, as we will see later, the variable u_0 will equate to the marginal return on capital.

Then ubd_j is the marginal return on deal *j* and this marginal return is essentially counted at the end of the year. Therefore the left-hand side can be considered the total risk-adjusted expected profit and it is equal to the total discounted marginal return on the deal. Later we will see another interpretation of this.

From equation (3) we see that we can also consider the $uz_{k,i}$ as probabilities. For simplicity of exposition we will assume that $k = 1$. Using the same idea as above, we will define the following:

$$Q_j = \sum_i pl_{i,j} \widetilde{uz}_i \qquad (19)$$

$$\text{with} \quad \widetilde{uz}_i = \frac{uz_i}{\sum_i uz_i}$$

and we will call Q_j the "risk adjusted expected losses impacting risk capital" .
From (3) we see that

$$\sum_i uz_i = \sum_i u_i - \sum_i \overline{\rho}_i = u_0 - 1 \qquad (20)$$

and so we can also write (19) as:

$$Q_j = \frac{\sum_i pl_{i,j} uz_i}{u_0 - 1} = \sum_i pl_{i,j} \widetilde{uz}_i \qquad (21)$$

The $\widetilde{uz}_i$ have interesting properties. They are probabilities in that they sum to one. They are the duals to the equations by scenario that impact the CVaR constraint to capital. They are zero for those scenarios that do not impact the capital constraint. If $Q_j = 0$, then increases in the deal *j* will not have any impact on the risk capital. If $Q_j < 0$, then increases in the deal *j* will require increases to risk capital and if $Q_j > 0$, then changes in the deal *j* will allow decreases to be made in the amount of risk capital required, everything else remaining the same. We will investigate Q_j more.

We observe the variable u_0 assuming that $\overline{rate} = 0$.

$$\begin{aligned} u_0 &= \sum_i u_i && \text{by (11)} \\ &= 1 + \sum_i \sum_k uz_{k,i} && \text{by (3)} \\ &= 1 + \sum_k \left(1 - \overline{c}_k\right) ucvar_k && \text{by (7)} \end{aligned} \qquad (22)$$

Now, consider the right-hand side of the *k* risk constraint and call it b_k .
Then:

(23) $$b_k = (1-\bar{c}_k)\overline{cvar_k} \times \overline{capital}$$

And assuming sufficient differentiability (or uniqueness of the dual solution), we have:

(24) $$\begin{aligned}\frac{\partial f}{\partial \overline{capital}} &= \sum_k \frac{\partial f}{\partial b_k}\frac{\partial b_k}{\partial \overline{capital}} \\ &= \sum_k ucvar_k (1-\bar{c}_k)\overline{cvar_k}\end{aligned}$$

with $f = \sum_{i \in I} \bar{\rho}_i \, funds1_i$ or the objective function of net return.

Now suppose we have only one *k* and that $\overline{cvar_k} = 1$, which means 100% loss of capital. Then we can substitute (22) into (24) and we have that:

(25) $$u_0 = 1 + \frac{\partial f}{\partial \overline{capital}}$$

This states that, in the case specified with one *k*, that u_0 is equal to the marginal return on capital.

Now if we have proper differentiability at the optimal solution, then we can write:

(26) $$\frac{\partial \overline{capital}}{\partial deal_j} = \frac{\dfrac{\partial f}{\partial deal_j}}{\dfrac{\partial f}{\partial \overline{capital}}}$$

In order to compute $\frac{\partial f}{\partial deal_j}$ in terms of its impact on the risk capital, we see how it behaves through the constraint associated with the uz_i dual variables. When considering $\frac{\partial f}{\partial deal_j}$ in this regard, we neglect the contribution from the bounds on the deals but we will return to this later. Thus we obtain the following:

(27) $$\frac{\partial f}{\partial deal_j} = -\sum_i pl_{i,j} uz_i = -(u_0 - 1)Q_j$$

And then we get:

(28)
$$\frac{\partial \overline{capital}}{\partial deal_j} = \frac{\dfrac{\partial f}{\partial deal_j}}{\dfrac{\partial f}{\partial \overline{capital}}} = \frac{-(u_0 - 1)Q_j}{u_0 - 1} = -Q_j$$

Therefore we can consider the contribution to risk capital, or *RC*, for a set of *j* deals as:

(29)
$$RC(J) = \sum_{j \in J} deal_j \frac{\partial \overline{capital}}{\partial deal_j} = -\sum_{j \in J} deal_j Q_j$$

That is, $-Q_j$ is the marginal change in capital per unit of deal *j*, or the marginal risk imposed by deal *j*. In using equation (29), we assume that the optimal solution behaves linearly around the optimal point. We can interpret $-deal_j Q_j$ as the contribution to the risk capital for deal *j*.

We can write the expected return in the following way with the simplifying assumptions mentioned before and omitting retrocessions.

(30)
$$\begin{aligned}
\sum_{i \in I} \bar{\rho}_i \, funds1_i &= \sum_{i \in I} \bar{\rho}_i \sum_j pl_{i,j} deal_j \\
&= \sum_{i \in I} (u_i - uz_i) \sum_j pl_{i,j} deal_j && \text{by (3)} \\
&= \sum_{i \in I} u_i \sum_j pl_{i,j} deal_j - \sum_{i \in I} uz_i \sum_j pl_{i,j} deal_j \\
&= \sum_j \left(ubd_j - \sum_{i \in I} uz_i \sum_j pl_{i,j} \right) deal_j && \text{by (4)} \\
&= \sum_j \left(u_0 R_j - (u_0 - 1) Q_j \right) deal_j && \text{by (18) and (21)}
\end{aligned}$$

If we interpret the risk adjusted expected profit for deal *j* as $u_0 R_j$ and the risk adjusted expected losses as $(u_0 - 1)Q_j$, then we could say: ***expected profit equals risk adjusted expected profit less risk adjusted expected losses at the margin.*** We explore this more in a subsequent paper.

Equation (30) shows that the expected profit is made up of two parts; $u_0 R_j$ and $(u_0 - 1)Q_j$. Suppose the first is non-zero and the latter is zero. This means that the deal is limited by an upper bound and that marginal increases in the deal will have no impact on the risk capital. This can happen because its losses all occur in scenarios where the corresponding $uz_i = 0$, or in scenarios that are not binding on the risk capital.

Suppose on the other hand that $(u_0 - 1)Q_j < 0$, then because of (28) and since $u_0 > 1$, this means that increases in the deal will require increases in the capital given everything else equal. This means there are losses in those scenarios where $uz_i > 0$. The opposite occurs when $(u_0 - 1)Q_j > 0$. Although this case is possible, the model would otherwise increase the deal as it wouldn't affect the risk capital and therefore it must be bounded above by a limit and so the term $u_0 R_j$ must be positive.

4. <u>Decomposing retrocessional cost</u>

Given this interpretation, it makes sense to allocate proportional to the contribution to risk or risk capital any returns due to retrocessional cover .

Consider the risk adjusted gains from a retrocessional cover. The contribution to risk capital from these gains should be applied against the marginal contribution to risk from the other covers.

Q_{retro} is the risk adjusted expected loss for the retro. Since we assume the sign is positive, the term amounts to contributions to risk capital, or how much additional risk capital is obtained. We naturally assume $Q_{retro} > 0$ and therefore ask how should the retro be apportioned to each deal. As we saw above, this means that the capital can be reduced. The natural answer is:

(31)
$$\{\text{proportional amount to } deal_j\} = \frac{Q_j deal_j}{\sum_l Q_l deal_l}$$

This measure gives full contribution to a recovery for a scenario that is more important or has a higher relative value for uz_i and these are the constraints that most affect the risk capital. The gains are apportioned with respect to those units where the corresponding scenario has an impact (see body of paper). Then the retrocessional costs (premiums) are allocated by the same proportions.

5. <u>Capital allocation</u>

Suppose we wanted to allocate capital. Following on from the previous section, one way to allocate capital is to do it so that the marginal contribution to return per unit of capital is the same for all units. We have a function for the contribution to risk capital summed over all deals given by (29).

Now suppose we have *k* sets $D_1, D_2, \ldots, D_k$ that represent a partition of the *j* deals; each one representing a separate unit. Let $cap_1, cap_2, \ldots, cap_k$ represent an allocation of capital, $\overline{capital}$, into the *k* units so that $\sum_{l=1}^{k} cap_l = \overline{capital}$. We want to pick the allocation so that the total marginal contribution to risk capital per unit of capital is the same for all units.

Therefore the allocation coefficient, $alloc_l$ for unit *l* is:

$$alloc_l = \overline{capital}\,\frac{RC(D_l)}{\sum_{m=1}^{k} RC(D_m)} = \overline{capital}\,\frac{\sum_{j\in D_l} deal_j Q_j}{\sum_j deal_j Q_j} \tag{32}$$

Then the total marginal contribution for all deals in a unit per unit of capital or:

$$\frac{-\sum_{j\in D_l} deal_j Q_j}{alloc_l} \tag{33}$$

is the same for all units and is equal to:

$$\frac{-\sum_j deal_j Q_j}{\overline{capital}} \tag{34}$$

which is itself the total marginal contribution of risk capital per unit of capital.

The contribution to risk capital can be used to compute the return on risk adjusted capital, RORAC, as in Theiler et. al. (2002) as:

$$RORAC = \frac{\sum_j \left(u_0 R_j - \left(u_0 - 1\right) Q_j\right) deal_j}{-\sum_j deal_j Q_j} \tag{35}$$

This is a simplified explanation. In a subsequent paper we will discuss non-unique solutions and what to do about them and zero marginal returns, reintroduce the risk free rate, discuss multiple risk constraints, and discuss how this applies to the Euler capital allocation principle.